First Order Differential Equations

A differential equation is an equation connecting x, y and the differential coefficients dy/dx, d^2y/dx^2, etc. For example,

$$y^2\frac{dy}{dx}+2xy=x$$

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+4y=\sin x$$

The order of a differential equation is the order of the differential coefficient of highest order in the equation. So the first equation above is of first order and the second is of second order.

First Order Differential Equations with Separable Variables.

If $\quad \frac{dy}{dx}=f(x)g(y)$

Then the solution is given by

$$\int\frac{1}{g(y)}dy=\int f(x)dx+c$$

provided that $\frac{1}{g(y)}$ can be integrated with respect to y and that f(x) can be integrated with respect to x.

eg1 Find the general solution of the equation

$$\frac{dy}{dx}=2xy \qquad (y>0)$$

eg2 Solve the equation

$$x\frac{dy}{dx}-xy=y \quad (x>0,\ y>0)$$

given that y = 1 when x = 1.

Exercise 5A Odds (Old Book)

or

Exercise 4A Evens (New Book)

eg1.

$$\frac{dy}{dx} = 2xy \qquad (y>0)$$

$$\frac{1}{y}\frac{dy}{dx} = 2x$$

$$\int \frac{1}{y}\frac{dy}{dx} \cdot dx = \int 2x \, dx$$

$$\int \frac{1}{y} \, dy = \int 2x \, dx$$

$$\ln \cancel{|}y\cancel{|} = x^2 + c$$ // not required because given $y>0$

$$y = e^{x^2+c}$$

$$y = e^{x^2} . e^{c}$$

$$y = Ae^{x^2}$$

eg2

$$x\frac{dy}{dx} - xy = y \qquad (x>0, y>0)$$

$$x\frac{dy}{dx} = y + xy$$

$$x\frac{dy}{dx} = y(1+x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1+x}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + 1.$$

$$\int \frac{1}{y}\frac{dy}{dx} \, dx = \int \frac{1}{x} + 1 \, dx$$

$$\int \frac{1}{y} \, dy = \int \frac{1}{x} + 1 \, dx$$

$$\ln y = \ln x + x + c$$

when $x=1, y=1$ $\quad \ln 1 = \ln 1 + 1 + c \implies c = -1.$

eg2 cont'd $\therefore$ $\ln y = \ln x + x - 1$

$$\ln \frac{y}{x} = x - 1$$

$$\frac{y}{x} = e^{x-1}$$

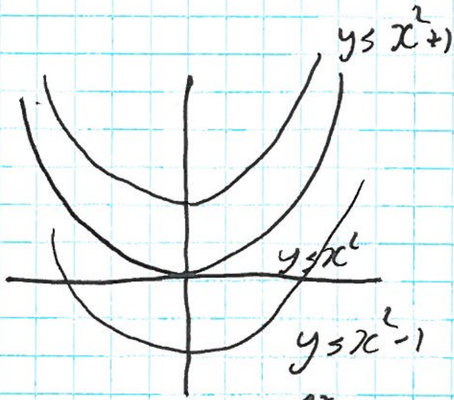
$$y = xe^{x-1}$$

Ex 4A

(1) $\frac{dy}{dx} = 2x$

$\int dy = \int 2x \, dx$

$y = x^2 + c$

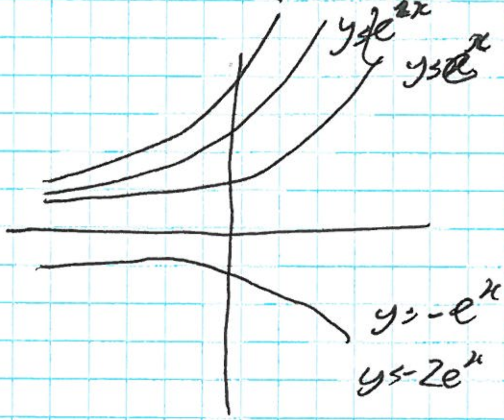


(2) $\frac{dy}{dx} = y$

$\int \frac{1}{y} dy = \int dx$

$\ln|y| = x + c$

$y = Ae^x$



(3) $\frac{dy}{dx} = x^2$

$\int dy = \int x^2 \, dx$

$y = \frac{1}{3}x^3 + c$

(4) $\frac{dy}{dx} = \frac{1}{x}$

$\int dy = \int \frac{1}{x} dx$

$y = \ln x + c$

if $c = \ln A$

$y = \ln x + \ln A$

$y = \ln Ax$

(5) $\frac{dy}{dx} = \frac{2y}{x}$

$\int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$

$\ln y = 2\ln x + c$

$\ln y - \ln x^2 = c$

$\ln \frac{y}{x^2} = c$

$\frac{y}{x^2} = e^c$

$y = Ax^2$

(6) $\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + c$

$y^2 = x^2 + 2c$

$y^2 - x^2 = 2c$

(7). $\frac{dy}{dx} = e^y$

$\int e^{-y}\,dy = \int dx$

$-e^{-y} = x + c$

$-\frac{1}{e^{+y}} = x + c$

$\cancel{e^y = \frac{1}{x+c}}$

$\frac{1}{e^y} = -x - c$

$e^y = \frac{1}{-x-c}$

$y = \ln\left|\frac{1}{-x-c}\right|$

(8) $\frac{dy}{dx} = \frac{y}{x(x+1)}$ ~~[illegible]~~

$\int \frac{1}{y} dy = \int \frac{1}{\cancel{x^2+x}} dx$

$\int \frac{1}{y} dy = \int \frac{A}{x} + \frac{B}{x+1} \, dx$

$\frac{A(x+1)+Bx}{x(x+1)}$

$\frac{Ax+A+Bx}{x(x+1)}$

$A+B=0$
$A=1$
$B=-1$

$\int \frac{1}{y} dy = \int \frac{1}{x} - \frac{1}{x+1} \, dx$

$\ln y = \ln x - \ln(x+1) + c$

$\ln y = \ln \frac{x}{x+1} + \ln A$

$\ln y = \ln \frac{Ax}{x+1}$

$y = \frac{Ax}{x+1}$

(9) $\frac{dy}{dx} = \cos x$

$\int dy = \int \cos x \, dx$

$y = \sin x + c$

(10) $\frac{dy}{dx} = y \cot x$

$\int \frac{1}{y} dy = \int \cot x \, dx$

$\ln y = \ln \sin x + c$

$\ln y = \ln A \sin x$

$y = A \sin x$

(11) $\frac{dy}{dt} = \sec^2 t$

$\int dy = \int \sec^2 t \, dt$

$y = \tan t + c$

(12) $\frac{dy}{dx} = x(1-x)$

$\int dy = \int x - x^2 \, dx$

$y = \frac{x^2}{2} - \frac{x^3}{3} + c$ Ans ??

(13) $\frac{dy}{dx} = \frac{y}{2x}$

$\int \frac{1}{y} dy = \int \frac{1}{2x} dx$

$\ln|y| = \frac{1}{2}\ln|x| + c$

$2\ln y = \ln x \cancel{+c}$

$\ln y^2 = \ln Ax$

$y^2 = \cancel{\tan} Ax$ if $A = 4a$ then $y^2 = 4ax$ As required

(14) $\frac{dy}{dx} = \frac{-xy}{9-x^2}$

$\int \frac{1}{\cancel{y}} dy = \int \frac{x}{x^2-9} dx$

$\int \frac{1}{y} dy = \int \frac{x}{(x+3)(x-3)} dx$ $\frac{A}{x+3} + \frac{B}{x-3}$

$\frac{A(x-3) + B(x+3)}{(x+3)(x-3)}$

$Ax - 3A + Bx + 3B \equiv x$

$A + B = 1$

$3B - 3A = 0$

$A = B = \frac{1}{2}$

(14) contd

$$\int \frac{1}{y}\, dy = \frac{1}{2}\int \frac{1}{x+3} + \frac{1}{x-3}\, dx$$

$$2\ln y = \ln(x+3) + \ln(x-3) + 2c$$

$$2\ln y = \ln(x^2-9) + 2c$$

$$\ln y^2 = \ln A(x^2-9)$$

$$y^2 = Ax^2 - 9A$$

$$y^2 - Ax^2 = -9A$$

if $k = -A$ $\quad y^2 + kx^2 = 9k$ As required

First Order Exact Differential Equations

Consider the differential equation

$$y + x\frac{dy}{dx} = 3x^4$$

This is a first order differential equation, however it is not possible to separate the variables.

From our earlier work on implicit differentiation however, we might recall that when we differentiate the product xy:

$\frac{d}{dx}(xy)$ let $u = x$ $v = y$

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = 1.\frac{dy}{dx}$

$\therefore \frac{d}{dx}(xy) = x\frac{dy}{dx} + y$ = LHS of diff eqⁿ.

This enables us to rewrite the original differential equation as:

$\frac{d}{dx}(xy) = 3x^4$

Integrating both sides $xy = \int 3x^4 dx$

$xy = \frac{3}{5}x^5 + c$

$y = \frac{3}{5}x^4 + \frac{c}{x}$

Equations of this form, where one side is an *exact* derivative of a product and the other side can be integrated with respect to x, are called *exact differential equations of the first order*.

These are often of the form $f(x)\frac{dy}{dx} + f'(x)y$, which can be written as $\frac{d}{dx}(f(x)y)$

eg3 Find the general solutions of the exact differential equations

(a) $\cos x\frac{dy}{dx} - y\sin x = x^2$ (b) $\frac{x}{y}\frac{dy}{dx} + \ln y = x + 1$

Exercise 4B NEW BOOK

eg3 (a) $\cos x \frac{dy}{dx} - y \sin x = x^2$

diff this to get that

check if unsure...

$y . -\sin x + \cos x . 1 \frac{dy}{dx}$

$\cos x \frac{dy}{dx} - y \sin x$ ✓

$$\frac{d}{dx}(y \cos x) = x^2$$

$$y \cos x = \int x^2 \, dx$$

$$y \cos x = \frac{x^3}{3} + c$$

$$y = \frac{1}{3} x^3 \sec x + c \sec x$$

(b) $\frac{x}{y} \frac{dy}{dx} + \ln y = x + 1$

check $x . \frac{1}{y} \frac{dy}{dx} + 1 . \ln y$ ✓

$$\frac{d}{dx}(x \ln y) = x + 1$$

$$x \ln y = \int x + 1 \, dx$$

$$x \ln y = \frac{x^2}{2} + x + c$$

$$\ln y = \frac{x}{2} + 1 + \frac{c}{x}$$

Ex4B

① $x\frac{dy}{dx} + y = \cos x$

$\frac{d}{dx}(xy) = \cos x$

$xy = \int \cos x \, dx$

$xy = \sin x + c$

$y = \frac{1}{x}\sin x + \frac{c}{x}$

② $e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{x}$

$\frac{d}{dx}(e^{-x}y) = xe^{x}$

$e^{-x}y = \int xe^{x} \, dx$ $\qquad v = x \quad \frac{du}{dx} = e^{x}$

$\frac{dv}{dx} = 1 \quad u = e^{x}$

$e^{-x}y = xe^{x} - \int e^{x} \, dx$

$e^{-x}y = xe^{x} - e^{x} + c$

$y = xe^{2x} - e^{2x} + ce^{x}$

③ $\sin x \frac{dy}{dx} + y\cos x = 3$

$\frac{d}{dx}(y\sin x) = 3$

$y\sin x = x + c$

$y = x\operatorname{cosec} x + c\operatorname{cosec} x$

④ $\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$

$\frac{d}{dx}\left(\frac{y}{x}\right) = e^x$

$\frac{y}{x} = \int e^x dx$

$\frac{y}{x} = e^x + c$

$y = xe^x + cx$

⑤ $\cancel{x^2 e^y dx}$ $x^2e^y\frac{dy}{dx} + 2xe^y = x$

$\frac{d}{dx}(x^2e^y)$

$x^2e^y\frac{dy}{dx} + e^y . 2x$

$\frac{d}{dx}(x^2e^y) = x$

$x^2e^y = \int x\, dx$

$x^2e^y = \frac{x^2}{2} + c$

$e^y = \frac{1}{2} + cx^{-2}$

$y = \ln\left[\frac{1}{2} + \frac{c}{x^2}\right]$

⑥ $4xy\frac{dy}{dx} + 2y^2 = x^2$

$\frac{d}{dx}(2xy^2)$

$2x . 2y\frac{dy}{dx} + 2y^2$

$\frac{d}{dx}(2xy^2) = x^2$

$2xy^2 = \int x^2 dx$

$2xy^2 = \frac{x^3}{3} + c$

$y^2 = \frac{x^2}{6} + \frac{c}{2x}$

$y = \pm\sqrt{\frac{x^2}{6} + \frac{c}{2x}}$

(7) (a) $x^2\frac{dy}{dx} + 2xy = 2x+1$

$\frac{d}{dx}(x^2y) = 2x+1$

$x^2y = \int 2x+1 \, dx$

$x^2y = x^2+x+c$

$y = 1+\frac{1}{x}+\frac{c}{x^2}$

(b) @ $(-\frac{1}{2},0)$ $\quad 0 = 1+\frac{1}{-\frac{1}{2}}+\frac{c}{(-\frac{1}{2})^2}$

$0 = 1-2+4c$

$c = \frac{1}{4}$

$\therefore y = 1+\frac{1}{x}+\frac{1}{4x^2}$

@ $(-\frac{1}{2},3)$ $\quad 3 = 1+\frac{1}{-\frac{1}{2}}+\frac{c}{(-\frac{1}{2})^2}$

$3 = 1-2+4c$

$c = 1$

$\therefore y = 1+\frac{1}{x}+\frac{1}{x^2}$

@ $(-\frac{1}{2},19)$ $\quad 19 = 1-2+4c$

$c = 5$

$\therefore y = 1+\frac{1}{x}+\frac{5}{x^2}$

(8) $\ln x \frac{dy}{dx} + \frac{y}{x} = \frac{1}{(x+1)(x+2)}$

$\cancel{\frac{d}{dx}(\ln x y}$

$$\frac{d}{dx}(y\ln x) = \int \frac{1}{(x+1)(x+2)} \, dx$$

$$y\ln x = \int \frac{A}{x+1} + \frac{B}{x+2} \, dx$$

$$y\ln x = \int \frac{1}{x+1} - \frac{1}{x+2} \, dx$$

$$y\ln x = \ln|x+1| - \ln|x+2| + c$$

$c = \ln A$

$$y\ln x = \ln \frac{A(x+1)}{x+2}$$

$$y = \frac{\ln\left|\frac{A(x+1)}{x+2}\right|}{\ln|x|}$$

(b) @ $(2,2)$ $\quad 2 = \frac{\ln\left|\frac{3A}{4}\right|}{\ln 2}$

$$2\ln 2 = \ln\left(\frac{3A}{4}\right)$$

$$\ln 4 = \ln\left(\frac{3A}{4}\right)$$

$\therefore \quad 4 = \frac{3A}{4}$

$\therefore \quad A = \frac{16}{3}$

$\therefore \quad y = \frac{\ln\left(\frac{16(x+1)}{3(x+2)}\right)}{\ln x}$

First Order Linear Differential Equations of the type $\frac{dy}{dx} + Py = Q$

Consider the solution of the differential equation $2ye^x \frac{dy}{dx} + y^2 e^x = e^{2x}$

$$\frac{d}{dx}(e^x y^2) = e^{2x}$$

$$e^x y^2 = \int e^{2x} dx$$

$$e^x y^2 = \frac{1}{2} e^{2x} + c \Rightarrow y^2 = \frac{1}{2} e^x + c e^{-x}$$

If we had tried to simplify the differential equation by dividing throughout by e^x, giving:

$$2y \frac{dy}{dx} + y^2 = e^x$$

In this form however, the LHS is not an exact derivative of a product. In fact, if we had been given the equation above, we would choose to multiply throughout by e^x in order to make the LHS exact in form.

The term e^x, which makes the LHS integrable, is called an *integrating factor*. When such an integrating factor can be determined, further differential equations can be solved. This is generalised below:

Consider the first order differential equation $\frac{dy}{dx} + Py = Q$

Multiplying throughout by Integrating Factor, I

$$I \frac{dy}{dx} + IPy = IQ$$

Compare LHS to exact derivative product

$$f(x) \frac{dy}{dx} + f'(x) y =$$

$$IP = f'(x) P \qquad IP = f'(x)$$

$$f(x) = f'(x) P \qquad I = f(x)$$

$$f(x)P = f'(x)$$

$$P = \frac{f(x)}{f'(x)} \qquad P = \frac{f'(x)}{f(x)}$$

Integrate wrt x

$$\int P dx = \int \frac{f'(x)}{f(x)} dx$$

$$\int P dx = \ln f(x)$$

$$f(x) = e^{\int P dx} = I$$

$$e^{\int P dx} \frac{dy}{dx} + P e^{\int P dx} y = e^{\int P dx} Q$$

$$\frac{d}{dx}\left(e^{\int P dx} y\right) = e^{\int P dx} Q$$

$$\boxed{e^{\int P dx} y = \int e^{\int P dx} Q \, dx}$$

In summary, the General Solution to the first order linear differential equation

$$\frac{dy}{dx} + Py = Q$$

is

$$ye^{\int P dx} = \int Qe^{\int P dx} + c$$

so long as $e^{\int P dx}$ can be found, and $Qe^{\int P dx}$ can be integrated.

<u>eg4</u> Find the general solution of the equation $\frac{dy}{dx} - y \tan x = 1$

<u>eg5</u> Find a suitable integrating factor and hence solve the differential equation

$$x\frac{dy}{dx} + 3y = \frac{e^x}{x^2}$$

<u>Exercise 4C</u> Pg 80 (New Book) Odd's

eg4 $\quad \frac{dy}{dx} - y\tan x = 1$

Compare with $\frac{dy}{dx} + Py = Q$

$\therefore P = -\tan x$

Now integrating factor $e^{\int P\,dx} = e^{\int -\tan x\,dx}$

$= e^{-\ln \sec x}$

$= e^{\ln(\sec x)^{-1}}$

$= (\sec x)^{-1}$

$= \cos x$

$\times$ by I.F.

$$\cos x \frac{dy}{dx} - \cos x . y\tan x = \cos x$$

$$\frac{d}{dx}(y\cos x) = \cos x$$

LHS here always becomes $y \times$ I.F.

RHS always becomes $Q \times$ I.F.

$$y\cos x = \int \cos x\,dx$$

$$y\cos x = \sin x + c$$

$$y = \tan x + c\sec x$$

eg5 $\quad x\frac{dy}{dx} + 3y = \frac{e^x}{x^2}$

Need to $\div x$ $\quad \frac{dy}{dx} + \frac{3}{x}y = \frac{e^x}{x^3}$

I.F. is $e^{\int 3/x\,dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

$\times$ I.F.

$$x^3\frac{dy}{dx} + x^3\frac{3}{x}y = x^3\frac{e^x}{x^3}$$

$$\frac{d}{dx}(x^3y) = e^x$$

$$x^3y = \int e^x\,dx$$

$$x^3y = e^x + c$$

$$y = \frac{e^x}{x^3} + \frac{c}{x^3}$$

Ex4C (New Book).

(1) $\frac{dy}{dx} + 2y = e^{x}$

Compare with $\frac{dy}{dx} + Py = Q$, $P = 2$

I.F. is $e^{\int P\,dx} = e^{\int 2\,dx} = e^{2x}$

Multiply throughout by I.F

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}.e^{x}$$

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{3x}$$

$$e^{2x}y = \int e^{3x}\,dx$$

$$e^{2x}y = \frac{1}{3}e^{3x} + c$$

$$y = \frac{1}{3}e^{x} + ce^{-2x}$$

(2) $\frac{dy}{dx} + y\cot x = 1$

I.F. $e^{\int \cot x\,dx} = e^{\ln|\sin x|} = \sin x$

$\therefore$ $\sin x\frac{dy}{dx} + \sin x.\cot x\, y = \sin x$

$$\frac{d}{dx}\left(y\sin x\right) = \sin x$$

$$y\sin x = \int \sin x\,dx$$

$$y\sin x = -\cos x + c$$

$$y = -\cot x + c\,\mathrm{cosec}\,x$$

(3) $$\frac{dy}{dx} + y\sin x = e^{\cos x}$$

I.F. is $$e^{\int \sin x \, dx} = e^{-\cos x}$$

$$\therefore \quad e^{-\cos x}\frac{dy}{dx} + y\sin x\, e^{-\cos x} = e^{0} \; 1$$

$$\frac{d}{dx}\left(ye^{-\cos x}\right) = 1$$

$$ye^{-\cos x} = \int 1\, dx$$

$$ye^{-\cos x} = x + c$$

$$y = xe^{\cos x} + ce^{\cos x}$$

(4) $$\frac{dy}{dx} - y = e^{2x}$$

I.F. is $$e^{\int -1\, dx} = e^{-x}$$

$$e^{-x}\frac{dy}{dx} - ye^{-x} = e^{-x}.e^{2x}$$

$$\frac{d}{dx}\left(ye^{-x}\right) = e^{x}$$

$$ye^{-x} = \int e^{x}\, dx$$

$$ye^{-x} = e^{x} + c$$

$$y = e^{2x} + ce^{x}$$

(5) $\frac{dy}{dx} + y\tan x = x\cos x$

I.F. is $e^{\int \tan x \, dx} = e^{\ln|\sec x|} = \sec x$

$\therefore \sec x \frac{dy}{dx} + y\sec x\tan x = x\cos x.\sec x$

$\frac{d}{dx}(y\sec x) = x$

$y\sec x = \int x\,dx$

$y\sec x = \frac{x^2}{2} + c$

$y = \frac{1}{2}x^2\cos x + c\cos x$

(6) $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$

I.F. is $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$\therefore x\frac{dy}{dx} + \frac{xy}{x} = \frac{x}{x^2}$

$\frac{d}{dx}(xy) = \frac{1}{x}$

$xy = \int \frac{1}{x}\,dx$

$xy = \ln x + c$

$y = \frac{1}{x}\ln x + \frac{c}{x}$

(7) $x^2\dfrac{dy}{dx} - xy = \dfrac{x^3}{x+2}$

$\div x^2$

$$\frac{dy}{dx} - \frac{y}{x} = \frac{x}{x+2}$$

I.F. is $e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \dfrac{1}{x}$

$\therefore \dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{1}{x}\cdot\dfrac{y}{x} = \dfrac{1}{x}\cdot\dfrac{x}{x+2}$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x+2}$$

$$\frac{y}{x} = \int \frac{1}{x+2}\,dx$$

$$y = x\ln|x+2| + cx$$

$x > -2$

$\therefore y = x\ln(x+2) + cx$

(8) $3x\dfrac{dy}{dx} + y = x$

$\div 3x$

$$\frac{dy}{dx} + \frac{y}{3x} = \frac{1}{3}$$

I.F. is $e^{\int \frac{1}{3x}dx} = e^{\frac{1}{3}\ln x} = e^{\ln x^{\frac{1}{3}}} = x^{\frac{1}{3}}$

$\therefore x^{\frac{1}{3}}\dfrac{dy}{dx} + x^{\frac{1}{3}}\cdot\dfrac{y}{3x} = \dfrac{1}{3}x^{\frac{1}{3}}$

$$\frac{d}{dx}\left(x^{\frac{1}{3}}y\right) = \frac{1}{3}x^{\frac{1}{3}}$$

$$x^{\frac{1}{3}}y = \int \frac{1}{3}x^{\frac{1}{3}}\,dx$$

$$x^{\frac{1}{3}}y = \frac{1}{3}\cdot\frac{3}{4}x^{\frac{4}{3}} + c$$

$$x^{\frac{1}{3}}y = \frac{1}{4}x^{\frac{4}{3}} + c$$

$$y = \frac{1}{4}x + cx^{-\frac{1}{3}}$$

(9) $(x+2)\frac{dy}{dx} - y = (x+2)$

$\div (x+2)$

$$\frac{dy}{dx} - \frac{y}{x+2} = 1$$

I.F. is $e^{\int -\frac{1}{x+2}dx} = e^{-\ln(x+2)} = e^{\ln(x+2)^{-1}} = \frac{1}{x+2}$

$$\frac{1}{x+2}\frac{dy}{dx} - \frac{1}{x+2}\cdot\frac{y}{(x+2)} = \frac{1}{x+2}$$

$$\frac{d}{dx}\left(\frac{y}{x+2}\right) = \frac{1}{x+2}$$

$$\frac{y}{x+2} = \int \frac{1}{x+2}\,dx$$

$$\frac{y}{x+2} = \ln(x+2) + c$$

$$y = (x+2)\ln(x+2) + c(x+2)$$

(10) $x\frac{dy}{dx} + 4y = \frac{e^x}{x^2}$

$\div x$

$$\frac{dy}{dx} + \frac{4}{x}y = \frac{e^x}{x^3}$$

I.F. is $e^{\int \frac{4}{x}dx} = e^{4\ln x} = e^{\ln x^4} = x^4$

$$\therefore\ x^4\frac{dy}{dx} + x^4\cdot\frac{4}{x}y = x^4\frac{e^x}{x^3}$$

$$\frac{d}{dx}(x^4y) = xe^x$$

$$x^4y = \int xe^x\,dx$$

(10) Cont'd

let $u = x^4$ $\quad \frac{dv}{dx} = e^x$

$\cancel{\frac{du}{dx} = 4x^3}$

$\frac{du}{dx} = 1 \quad V = e^x$

$\therefore \quad x^4 y = xe^x - \int e^x dx$

$x^4 y = xe^x - e^x + c$

$y = \frac{e^x}{x^3} - \frac{e^x}{x^4} + \frac{c}{x^4}$

(11) $x \frac{dy}{dx} + 2y = e^x$

$\div x$

$\frac{dy}{dx} + \frac{2y}{x} = \frac{e^x}{x}$

I.F. is $e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$\therefore \quad x^2 \frac{dy}{dx} \quad x^2 \frac{2y}{x} = xe^x$

$\frac{d}{dx}(x^2 y) = xe^x$

$x^2 y = \int xe^x dx$ $\quad u = x \quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1 \quad v = e^x$

$x^2 y = xe^x - \int e^x dx$

$x^2 y = xe^x - e^x + c$

when $x = 1, y = 1$

$1 = e^1 - e^1 + c \Rightarrow c = 1$

$\therefore \quad x^2 y = xe^x - e^x + 1$

$y = \frac{e^x}{x} - \frac{e^x}{x^2} + \frac{1}{x^2}$

(12) $x^3\frac{dy}{dx} - x^2y = 1$

$\div x^3$

$$\frac{dy}{dx} - \frac{y}{x} = \frac{1}{x^3}$$

I.F. is $e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$

$\therefore \quad \frac{1}{x}\frac{dy}{dx} - \frac{1}{x}\frac{y}{x} = \frac{1}{x^4}$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x^{-4}$$

$$\frac{y}{x} = \int x^{-4}\,dx$$

$$\frac{y}{x} = -\frac{1}{3}x^{-3} + c$$

when $x=1, y=1$

$$1 = -\frac{1}{3} + c \Rightarrow c = \frac{4}{3}$$

$$\frac{y}{x} = -\frac{1}{3x^3} + \frac{4}{3}$$

$$y = \frac{4x}{3} - \frac{1}{3x^2}$$

(13) (a) $\left(x+\frac{1}{x}\right)\frac{dy}{dx} + 2y = 2(x^2+1)^2$

$$\left(\frac{x^2+1}{x}\right)\frac{dy}{dx} + 2y = 2(x^2+1)^2$$

$\cancel{\div\left(\frac{x^2+1}{x}\right) \quad \frac{dy}{dx} + 2xy}$

$\div\left(\frac{x^2+1}{x}\right) \quad \frac{dy}{dx} + \frac{2x}{x^2+1}y = 2(x^2+1)^2\cdot\frac{x}{(x^2+1)} = 2x(x^2+1)$

13 (a) contd. $\frac{dy}{dx} + \frac{2xy}{x^2+1} = 2x^3 + 2x$

I.F. is $e^{\int \frac{2x}{x^2+1}dx}$ let $u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$ $\int \frac{\cancel{2x}}{u} \cdot \frac{du}{\cancel{2x}} = \ln(x^2+1)$

$e^{\ln(x^2+1)} = x^2+1$

$\times$ I.F.

$\therefore (x^2+1)\frac{dy}{dx} + (x^2+1)\frac{2xy}{x^2+1} = (x^2+1)2x(x^2+1)$

$\frac{d}{dx}(y(x^2+1)) = 2x(x^2+1)^2$

$y(x^2+1) = \int 2x(x^2+1)^2 \, dx$ $u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$y(x^2+1) = \int \cancel{2x} u^2 \cdot \frac{du}{\cancel{2x}}$

$y(x^2+1) = \int u^2 \, du$

$y(x^2+1) = \frac{1}{3}(x^2+1)^3 + c$

$y = \frac{1}{3}(x^2+1)^2 + \frac{c}{x^2+1}$

(b) @ $x=1, y=1$

$1 = \frac{4}{3} + \frac{c}{2}$

$c = -\frac{2}{3}$

$\therefore y = \frac{1}{3}(x^2+1)^2 - \frac{2}{3(x^2+1)}$

(14) (a) $\cos x \frac{dy}{dx} + y = 1$

$\div \cos x$

$$\frac{dy}{dx} + y\sec x = \sec x$$

I.F. is $e^{\int \sec x\, dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$

$\times$ I.F.

$$(\sec x + \tan x)\frac{dy}{dx} + (\sec x + \tan x)y\sec x = \sec^2 x + \sec x\tan x$$

$$\frac{d}{dx}\left(y(\sec x + \tan x)\right) = \int \sec^2 x + \sec x\tan x \, dx$$

$$y(\sec x + \tan x) = \tan x + \sec x + c$$

$$y = 1 + \frac{c}{\sec x + \tan x}$$

(b) when $x = 0$ $y = 2$

$$2 = 1 + \frac{c}{\sec 0 + \tan 0}$$

$$c = 1$$

$$y = 1 + \frac{1}{(\sec x + \tan x)}$$

Solving differential equations using a change of variable

Sometimes, a differential equation that cannot easily be integrated can be transformed into one which we know how to solve by using a substitution which changes one of the variables in the equation into a new variable.

eg6 Using the substitution $z = \dfrac{y}{x}$, transform the equation $x\dfrac{dy}{dx} = x + y$, x >0, into a differential equation in z and x. By first solving this new differential equation, find the general solution of the original equation.

eg7 Use the substitution $z = y^{-2}$ to find the general solution of the differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^3$

Find also y in terms of x, given that y = 1 at x = 1.

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eg6 $\quad z = \dfrac{y}{x}$

$$x\frac{dy}{dx} = x + y \quad \text{—} \ (1)$$

* This could be solved with an I.F. *

$$x\frac{dy}{dx} - y = x$$

$$\frac{dy}{dx} - \frac{1}{x}y = 1$$

$$IF = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x}$$

$$\frac{y}{x} = \int \frac{1}{x}\,dx$$

$$\frac{y}{x} = \ln x + \cancel{k}$$

$$\frac{y}{x} = \ln Ax$$

$$y = x\ln Ax$$

$$y = zx$$

Use product rule, implicitly to find $\dfrac{dy}{dx}$

$$y = zx$$

$$u = z \qquad v = x$$

$$\frac{du}{dx} = 1\frac{dz}{dx} \qquad \frac{dv}{dx} = 1$$

$$\therefore \frac{dy}{dx} = z + x\frac{dz}{dx}$$

eliminate, by substitution in (1)

$$x\left(z + x\frac{dz}{dx}\right) = x + xz$$

$$xz + x^2\frac{dz}{dx} = x + xz$$

$$x^2\frac{dz}{dx} = x$$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\int dz = \int \frac{1}{x}\,dx$$

$$z = \ln|x| + c \qquad \text{let } c = \ln A \text{ and } x > 0$$

$$z = \ln x + \ln A$$

$$z = \ln Ax$$

Now $\quad \dfrac{y}{x} = \ln Ax$

$$\therefore y = x\ln Ax$$

eg 7

$$z = y^{-2} \qquad \frac{dy}{dx} + \frac{y}{x} = x^2 y^3 \quad \text{— (1)}$$

$$\cancel{z^2} = y$$

$$z = \frac{1}{y^2}$$

$$y^2 = \frac{1}{z}$$

$$y = z^{-1/2}$$

$$\frac{dy}{dx} = -\frac{1}{2} z^{-3/2} \frac{dz}{dx}$$

In (1)

$$-\frac{1}{2} z^{-3/2} \frac{dz}{dx} + \frac{z^{-1/2}}{x} = x^2 \left(z^{-1/2}\right)^3$$

$$-\frac{1}{2} z^{-3/2} \frac{dz}{dx} + \frac{z^{-1/2}}{x} = x^2 z^{-3/2}$$

$\div z^{-3/2}$

$$-\frac{1}{2} \frac{dz}{dx} + \frac{z^{-1/2}}{x z^{-3/2}} = x^2$$

$$-\frac{1}{2} \frac{dz}{dx} + \frac{z^{-1/2 - -3/2}}{x} = x^2$$

$\times -2$

$$\frac{dz}{dx} - \frac{2}{x} z = -2x^2$$

$$\text{Now Int Fact} = e^{-\int \frac{2}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

$$\therefore \quad x^{-2} \frac{dz}{dx} - e^{-2} \cdot \frac{2}{x} z = -2x^2 \cdot \cancel{x}^{-2}$$

$$\frac{d}{dx}\left(x^{-2} z\right) = -2$$

$$x^{-2} z = \int -2 \, dx$$

$$x^{-2} z = -2x + c$$

eq7 cont'd

$$Z = -2x^{3} + cx^{2}$$

$$\frac{1}{y^2} = cx^2 - 2x^3$$

$$y^2 = \frac{1}{cx^2 - 2x^3}$$

$$y^2 = \frac{1}{x^2(c-2x)}$$

$$y = \frac{1}{x(c-2x)^{1/2}}$$

at $y=1, x=1$

$$1 = \frac{1}{1(c-2)^{1/2}}$$

$$(c-2)^{1/2} = 1$$

$$c-2 = 1$$

$$c = 3$$

$$\therefore \quad y = \frac{1}{x(3-2x)^{1/2}}$$

Ex4D

(1) $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$ — (1)

if $z = \frac{y}{x}$ then $y = xz$

$u = x$ $\quad$ $v = z$

$\frac{du}{dx} = 1$ $\quad$ $\frac{dv}{dx} = \frac{dz}{dx}$

$\frac{dy}{dx} = x\frac{dz}{dx} + z$

in (1) $x\frac{dz}{dx} + z = z + \frac{1}{z}$

$x\frac{dz}{dx} = \frac{1}{z}$

$z\,dz = \frac{1}{x}\,dx$

$\frac{z^2}{2} = \ln x + c$

$z^2 = 2(\ln x + c)$

$\frac{y^2}{x^2} = 2(\ln x + c)$

$y^2 = 2x^2(\ln x + c)$

(2) $\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2}$ — (1)

if $z = \frac{y}{x}$ then $y = xz$ $\quad$ $\frac{dy}{dx} = x\frac{dz}{dx} + z$

in (1) $x\frac{dz}{dx} + z = z + \left(\frac{1}{z^2}\right)$

$x\frac{dz}{dx} = \frac{1}{z^2}$

$\int z^2\,dz = \int \frac{1}{x}\,dx$

② contd $\frac{z^3}{3} = \ln x + c$

$$z^3 = 3(\ln x + c)$$

$$\frac{y^3}{x^3} = 3(\ln x + c)$$

$$y^3 = 3x^3(\ln x + c)$$

③ $\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$ — ①

If $z = \frac{y}{x}$ then $y = xz$ $\quad \frac{dy}{dx} = x\frac{dz}{dx} + z$

In ① $x\frac{dz}{dx} + z = z + z^2$

$$x\frac{dz}{dx} = z^2$$

$$\int z^{-2}\,dz = \int \frac{1}{x}\,dx$$

$$-\frac{1}{z} = \ln x + c$$

$$-\frac{x}{y} = \ln x + c$$

$$y = \frac{-x}{\ln x + c}$$

(4) $\frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2}$ — (1)

If $z = \frac{y}{x} \Rightarrow y = xz \Rightarrow \frac{dy}{dx} = x\frac{dz}{dx} + z$

in (1) $x\frac{dz}{dx} + z = \frac{x^3 + 4x^3z^3}{3x^3z^2}$

$x\frac{dz}{dx} + z = \frac{1+4z^3}{3z^2}$

$x\frac{dz}{dx} = \frac{1+4z^3}{3z^2} - z$

$x\frac{dz}{dx} = \frac{1+4z^3-3z^3}{3z^2}$

$x\frac{dz}{dx} = \frac{1+z^3}{3z^2}$

$\int \frac{3z^2}{1+z^3}\,dz = \int \frac{1}{x}\,dx$

$\ln(1+z^3) = \ln x + c$

$1+z^3 = e^{\ln x + c}$

$1+z^3 = e^{\ln x} \cdot e^c$

$1+z^3 = Ax$

$z^3 = Ax - 1$

$\frac{y^3}{x^3} = Ax - 1$

$y^3 = x^3(Ax-1)$

(5) $z = y^{-2}$ $\qquad \frac{dy}{dx} + \left(\frac{1}{2}\tan x\right)y = -(2\sec x)y^3$ — (1)

$z = \frac{1}{y^2}$

$y^2 = \frac{1}{z}$

$y^2 = z^{-1}$

$y = z^{-1/2} \implies y^3 = z^{-3/2}$

$\frac{dy}{dx} = -\frac{1}{2}z^{-3/2}\frac{dz}{dx}$

in (1) $\quad -\frac{1}{2}z^{-3/2}\frac{dz}{dx} + \left(\frac{1}{2}\tan x\right)z^{-1/2} = -(2\sec x)z^{-3/2}$

$\cancel{\div z^{-1/2} \quad -\frac{1}{2}z^{3}\frac{dz}{dx} + \left(\frac{1}{2}\tan x\right) = -(2\sec x)z^3}$

$\cancel{\times -2 \quad z^{3}\frac{dz}{dx} - \tan x = 4\sec x\, z^3}$

$\cancel{\div z^{3} \quad \frac{dz}{dx} - z^{-3}\tan x = 4\sec x}$

$\div z^{-3/2} \quad -\frac{1}{2}\frac{dz}{dx} + \left(\frac{1}{2}\tan x\right)\frac{z^{-1/2}}{z^{-3/2}} = -(2\sec x)$

$-\frac{1}{2}\frac{dz}{dx} + \left(\frac{1}{2}\tan x\right)z = -2\sec x$

$\times -2 \quad \frac{dz}{dx} - z\tan x = 4\sec x$

Int fact is $e^{\int -\tan x\,dx} = e^{-\ln\sec x} = e^{\ln\cos x} = \cos x$

$\therefore \cos x\frac{dz}{dx} - \cos x\tan x\, z = 4\sec x . \cos x$

$\frac{d}{dx}(z\cos x) = 4$

$z\cos x = \int 4\,dx$

(5) contd $z\cos x = 4x + c$

$$z = \frac{4x+c}{\cos x}$$

$$\frac{1}{y^2} = \frac{4x+c}{\cos x}$$

$$y^2 = \frac{\cos x}{4x+c}$$

$$y = \sqrt{\frac{\cos x}{4x+c}}$$

(6) $z = x^{\frac{1}{2}}$ $\qquad \frac{dx}{dt} + t^2 x = t^2 x^{\frac{1}{2}}$ — (1)

$x = z^2$

$$\frac{dx}{dt} = 2z\frac{dz}{dt}$$

in (1) $\quad 2z\frac{dz}{dt} + t^2 z^2 = t^2 z$

$\div 2z$

$$\frac{dz}{dt} + \frac{t^2 z}{2} = \frac{t^2}{2}$$

Int fact $\quad e^{\int \frac{t^2}{2} dt} = e^{\frac{t^3}{6}}$

$$\therefore e^{\frac{t^3}{6}}\frac{dz}{dt} + e^{\frac{t^3}{6}}\cdot\frac{t^2}{2}z = e^{\frac{t^3}{6}}\cdot\frac{t^2}{2}$$

$$\frac{d}{dt}\left(e^{\frac{t^3}{6}}z\right) = \frac{t^2}{2}e^{\frac{t^3}{6}}$$

$$e^{\frac{t^3}{6}}z = \frac{1}{2}\int t^2 e^{\frac{t^3}{6}}\,dt$$

let $u = \frac{t^3}{6}$

$$\frac{du}{dt} = \frac{t^2}{2}$$

$$dt = \frac{2\,du}{t^2}$$

(6) contd

$$e^{t^3/6} z = \frac{1}{2}\int t^2 e^u \cdot 2\frac{du}{t^2}$$

$$e^{t^3/6} z = \int e^u du$$

$$e^{t^3/6} z = e^{t^3/6} + c$$

$$z = 1 + ce^{-t^3/6}$$

Now

$$x^{1/2} = 1 + ce^{-t^3/6}$$

$$\therefore \quad x = (1 + ce^{-t^3/6})^2$$

(7). $z = y^{-1}$ $\qquad\qquad \frac{dy}{dx} - \frac{1}{x}y = \frac{(x+1)^3}{x}y^2 \quad$ — (1)

$$y = z^{-1}$$

$$\frac{dy}{dx} = -z^{-2}\frac{dz}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

in (1) $\quad -\frac{1}{z^2}\frac{dz}{dx} - \frac{1}{x}\cdot\frac{1}{z} = \frac{(x+1)^3}{x}\cdot\frac{1}{z^2}$

$x -z^2$

$$\frac{dz}{dx} + \frac{1}{x}z = -\frac{(x+1)^3}{x}$$

Int fact $\quad e^{\int 1/x \, dx} = e^{\ln x} = x$

$$\therefore \quad x\frac{dz}{dx} + x\cdot\frac{1}{x}z = -x\frac{(x+1)^3}{x}$$

$$\frac{d}{dx}(xz) = -(x+1)^3$$

$$xz = -\int (x+1)^3 dx$$

let $u = x+1$

$\frac{du}{dx} = 1$

$dx = du$

(7) contd

$$xz = -\int u^3 \, du$$

$$xz = -\frac{1}{4}u^4 + c$$

$$xz = -\frac{1}{4}(x+1)^4 + c$$

$$xz = \frac{4c - (x+1)^4}{4}$$

$$z = \frac{4c - (x+1)^4}{4x}$$

$$\frac{1}{y} = \frac{4c - (x+1)^4}{4x}$$

$$\therefore \quad y = \frac{4x}{4c - (x+1)^4}$$

(8) (a) $z = y^2$ $\qquad\qquad 2(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{y} \quad -①$

$$y = z^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx}$$

in ① $\quad \cancel{2}(1+x^2) \cdot \frac{1}{\cancel{2}} z^{-\frac{1}{2}} \frac{dz}{dx} + 2x z^{\frac{1}{2}} = z^{-\frac{1}{2}}$

$\div z^{-\frac{1}{2}}$

$$(1+x^2)\frac{dz}{dx} + \frac{2x z^{\frac{1}{2}}}{z^{-\frac{1}{2}}} = 1$$

$$(1+x^2)\frac{dz}{dx} + 2xz = 1$$

$\div (1+x^2)$

$$\frac{dz}{dx} + \frac{2x}{(1+x^2)} z = \frac{1}{1+x^2}$$

(8) contd Int Fact $e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = (1+x^2)$

$$\therefore (1+x^2) + (1+x^2) \cdot \frac{2x}{(1+x^2)} z = (1+x^2) \times \frac{1}{(1+x^2)}$$

$$\frac{d}{dx}\left((1+x^2)z\right) = 1$$

$$(1+x^2)z = \int 1 \, dx$$

$$(1+x^2)z = x + c$$

$$z = \frac{x+c}{1+x^2}$$

$$y^2 = \frac{x+c}{1+x^2}$$

$$y = \sqrt{\frac{x+c}{1+x^2}}$$

(b) @ (0,2) $\quad 2 = \sqrt{\frac{c}{1}}$

$$\therefore c = 4$$

$$\therefore y = \sqrt{\frac{x+4}{1+x^2}}$$

(9) $Z = y^{-(n-1)}$

$y = Z^{-\frac{1}{n-1}}$

$\frac{dy}{dx} = -\frac{1}{(n-1)} Z^{\left(-\frac{1}{n-1}-1\right)} \frac{dz}{dx} = -\frac{1}{n-1} Z^{\left(\frac{-1-(n-1)}{n-1}\right)} \cdot \frac{dz}{dx} = -\frac{1}{n-1} Z^{-\frac{n}{n-1}} \cdot \frac{dz}{dx}$

$\therefore \frac{dy}{dx} + Py = Qy^n$

becomes $-\frac{1}{n-1} Z^{-\frac{n}{n-1}} \frac{dz}{dx} + P Z^{-\frac{1}{n-1}} = Q\left(Z^{-\frac{1}{n-1}}\right)^n$

$\div Z^{-\frac{n}{n-1}}$

$-\frac{1}{n-1} \frac{dz}{dx} + P \frac{Z^{-\frac{1}{n-1}}}{Z^{-\frac{n}{n-1}}} = Q$

$\cdot \ -\frac{1}{n-1} \frac{dz}{dx} + P Z^{-\frac{1}{n-1}+\frac{n}{n-1}} = Q$

$-\frac{1}{n-1} \frac{dz}{dx} + P Z^{\frac{n-1}{n-1}} = Q$

$\times -(n-1)$

$\frac{dz}{dx} - P(n-1)Z = -Q(n-1)$ As required!

(10) $u = y + 2x$ $\qquad\qquad \dfrac{dy}{dx} = -\left(\dfrac{1+2y+4x}{1+y+2x}\right)$ — (1)

$y = u - 2x$

$\dfrac{dy}{dx} = 1.\dfrac{du}{dx} - 2$

In (1) $\dfrac{du}{dx} - 2 = -\left(\dfrac{1+2(u-2x)+4x}{1+u-2x+2x}\right)$

$\dfrac{du}{dx} - 2 = -\left(\dfrac{1+2u-4x+4x}{1+u}\right)$

$\dfrac{du}{dx} - 2 = -\left(\dfrac{1+2u}{1+u}\right)$

$\dfrac{du}{dx} = 2 - \dfrac{(1+2u)}{1+u}$

$\dfrac{du}{dx} = \dfrac{2(1+u) - (1+2u)}{1+u}$

$\dfrac{du}{dx} = \dfrac{2+2u-1-2u}{1+u}$

$\dfrac{du}{dx} = \dfrac{1}{1+u}$

$\int 1+u \, du = \int dx$

$u + \dfrac{u^2}{2} = x + c$

$2u + u^2 = 2x + 2c$

$u(2+u) = 2x + 2c$

$\therefore (y+2x)(2+y+2x) = 2x + 2c$

$2y + y^2 + 2xy + 4x + 2xy + 4x^2 - 2x = 2c$ *

$4x^2 + 4xy + 2y + 2x = 2c = k$ As required