1. Probability

A2 Conditional Probability

Conditional probability is the probability of an event occurring ***given that*** another event has occurred.

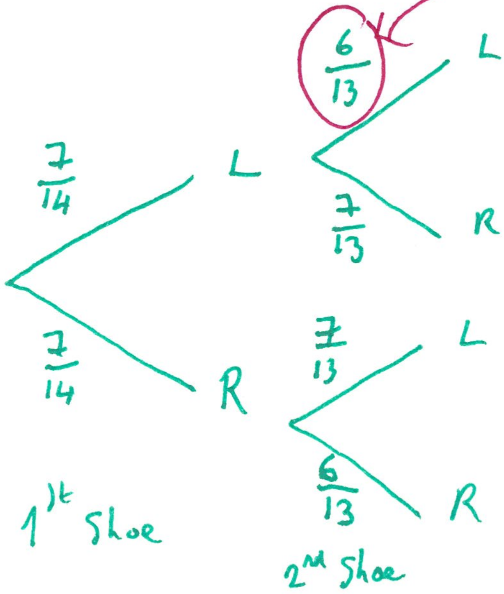
Consider the following GCSE question:

A clown has seven pairs of shoes, one pair in each of the colours of the rainbow. The shoes are kept in a trunk in a dark room. The clown selects two shoes at random.

a. What is the probability that the clown selects one left shoe and one right shoe?
b. What is the probability of selecting a matching pair of shoes?

This is best approached using a tree diagram

outcome is conditional on the First Shoe



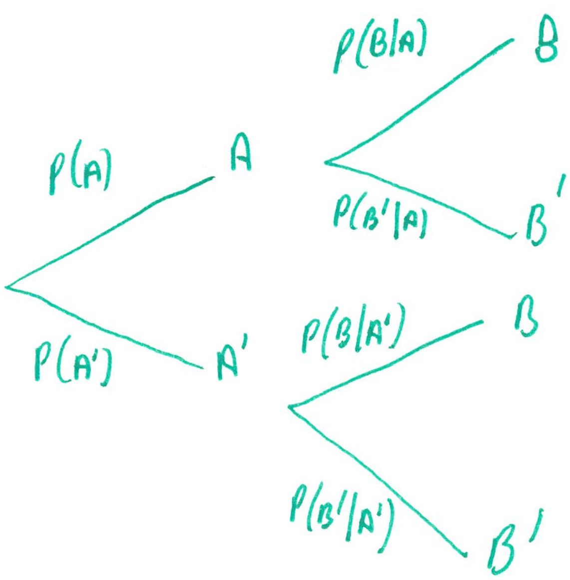
1a) P(one left, one right)

$= P(L \cap R) + P(R \cap L)$

$= \left(\frac{7}{14} \times \frac{7}{13}\right) + \left(\frac{7}{14} \times \frac{7}{13}\right) = 0.538$

$= \frac{7}{13}\left(\frac{7}{14} + \frac{7}{14}\right)$

In general, for two events, A and B, the probability of B occurring given that A has already occurred is written as P(B|A). This can be usefully represented on a tree diagram:



$P(A \cap B) = P(A)P(B|A)$

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

$P(A \cap B') = P(A)P(B'|A)$

$P(A' \cap B) = P(A')P(B|A')$

$P(A' \cap B') = P(A')P(B'|A')$

This leads to the conditional probability formula for dependent events

$$P(A \cap B) = P(A)P(B|A)$$

which is often more usefully written

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

it also holds that

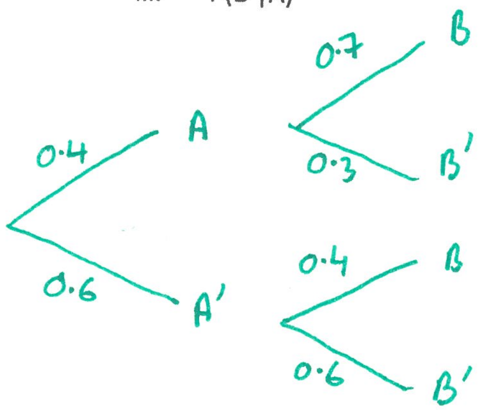
$$P(A \cap B) = P(B)P(A|B)$$

Depending on the nature of the information you are given, problems involving conditional probability can be supported by drawing tree diagrams, Venn diagrams or two-way tables. We need to become familiar with each.

Tree Diagrams

Eg4 Two events A and B occur such that $P(A') = 0.6$, $P(B|A) = 0.7$, $P(B|A') = 0.4$
(a) Represent this information on a tree diagram
(b) Use your tree diagram to find
i. $P(A \cap B)$
ii. $P(A' \cap B')$
iii. $P(B'|A)$

(a)



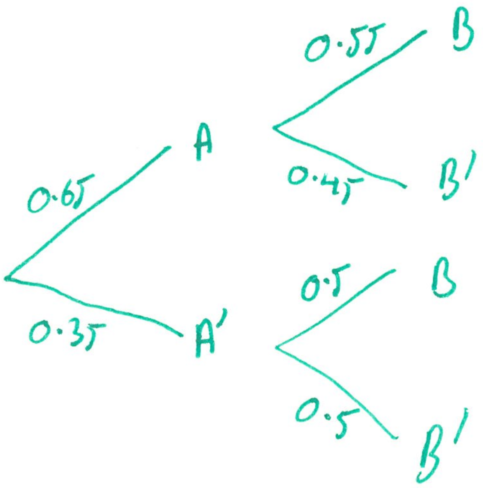
(b)(i) $P(A \cap B) = 0.4 \times 0.7 = 0.28$

(ii) $P(A' \cap B') = 0.6 \times 0.6 = 0.36$

(iii) $P(B'|A) = 0.3$

Eg5 Two events A and B occur such that $P(B|A) = 0.55$, $P(B|A') = 0.5$ and $P(A) = 0.65$
(a) Represent this information on a tree diagram
(b) Use your tree diagram to find
i. $P(A \cap B)$
ii. $P(B)$
iii. $P(A|B)$

(a)



(b)(i) $P(A \cap B) = 0.65 \times 0.55 = 0.3575$

(ii) $P(B) = (0.65 \times 0.55) + (0.35 \times 0.5)$

$= 0.5325$

(iii) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3575}{0.5325}$

$= 0.6714$

Eg6 The turnout of spectators at a motor rally is dependent on the weather. On a rainy day the probability of a big turnout is 0.4, but if it does not rain, the probability of a big turnout increases to 0.9. The weather forecast gives a probability of 0.75 that it will rain on the day of the race.

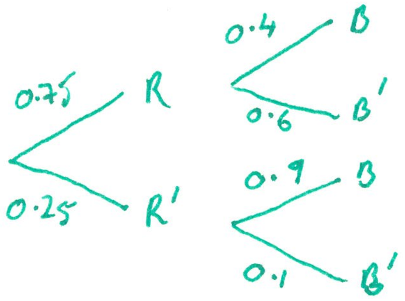
(a) Draw a tree diagram to represent this information.

(b) Find the probability that

i. there is a big turnout and it rains

ii. there is a big turnout

(a)



(b) (i) $P(B \cap R) = 0.75 \times 0.4 = 0.3$

(ii) $P(B) = P(B \cap R) + P(B \cap R')$

$= 0.3 + (0.25 \times 0.9)$

$= 0.525$

Exercise

1. A and B are two events such that $P(A|B) = 0.1$, $P(A|B') = 0.6$ and $P(B) = 0.3$

   (a) Draw a tree diagram to represent this information

   (b) Find

   i. $P(A \cap B)$

   ii. $P(A \cap B')$

   iii. $P(A)$

   iv. $P(B|A)$

   v. $P(B|A')$

2. A bag contains five red and four blue tokens. A token is chosen at random, the colour recorded and the token is not replaced. A second token is chosen and the colour recorded. Find the probability that

   a the second token is red given the first token is blue,

   b the second token is blue given the first token is red,

   c both tokens chosen are blue,

   d one red token and one blue token are chosen.

3. A box of 24 chocolates contains 10 dark and 14 milk chocolates. Linda chooses a chocolate at random and eats it, followed by another one. Find the probability that Linda eats

   a two dark chocolates,

   b one dark and one milk chocolate.

4. Jean always goes to work by bus or takes a taxi. If one day she goes to work by bus, the probability she goes to work by taxi the next day is 0.4. If one day she goes to work by taxi, the probability she goes to work by bus the next day is 0.7.

   Given that Jean takes the bus to work on Monday, find the probability that she takes a taxi to work on Wednesday.

5. Sue has two coins. One is fair, with a head on one side and a tail on the other.

The second is a trick coin and has a tail on both sides. Sue picks up one of the coins at random and flips it.

a Find the probability that it lands heads up.

b Given that it lands tails up, find the probability that she picked up the fair coin.

6. In a factory, machines *A*, *B* and *C* produce electronic components. Machine *A* produces 16% of the components, machine *B* produces 50% of the components and machine *C* produces the rest. Some of the components are defective. Machine *A* produces 4%, machine *B* 3% and machine *C* 7% defective components.

a Draw a tree diagram to represent this information.

Find the probability that a randomly selected component is

b produced by machine *B* and is defective,

c defective.

Given that a randomly selected component is defective,

d find the probability that it was produced by machine *B*.

7. A garage sells three types of fuel; U95, U98 and diesel. In a survey of 200 motorists buying fuel at the garage, 80 are female and the rest are male. Of the 90 motorists buying 'U95' fuel, 50 were female and of the 70 motorists buying diesel, 60 were male. A motorist does not buy more than one type of fuel.

Find the probability that a randomly chosen motorist

a buys U98 fuel,

b is male, given that the motorist buys U98 fuel.

Garage records indicate that 10% of the motorists buying U95 fuel, 30% of the motorists buying U98 fuel and 40% of the motorists buying diesel have their car serviced by the garage.

A motorist is chosen at random.

c Find the probability that this motorist has his or her car serviced by the garage.

d Given the motorist has his or her car serviced by the garage, find the probability that the motorist buys diesel fuel.

Numerical Answers

(1b) (i) 0.03 (ii) 0.42 (iii) 0.45 (iv) 1/15 (v) 27/55 (2a) 5/8 (b) ½ (c) 1/6 (d) 5/9

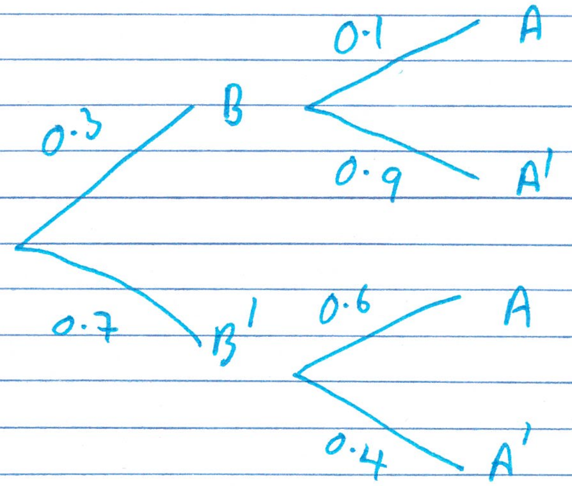
(3a) 15/92 (b) 35/69 (4) 0.36 (5a) ¼ (b) 1/3

(6b) 0.015 (c) 0.0452 (d) 0.332 (7a) 0.2 (b) 0.5 (c) 49/200 (d) 4/7

Exercise

(1) $P(A|B) = 0.1$ $\quad P(A|B') = 0.6$ $\quad P(B) = 0.3$

each measure has B as first event



(i) $P(A \cap B) = 0.3 \times 0.1 = 0.03$

(ii) $P(A \cap B') = 0.7 \times 0.6 = 0.42$

(iii) $P(A) = P(B \cap A) + P(B' \cap A)$

$= 0.03 + 0.42$

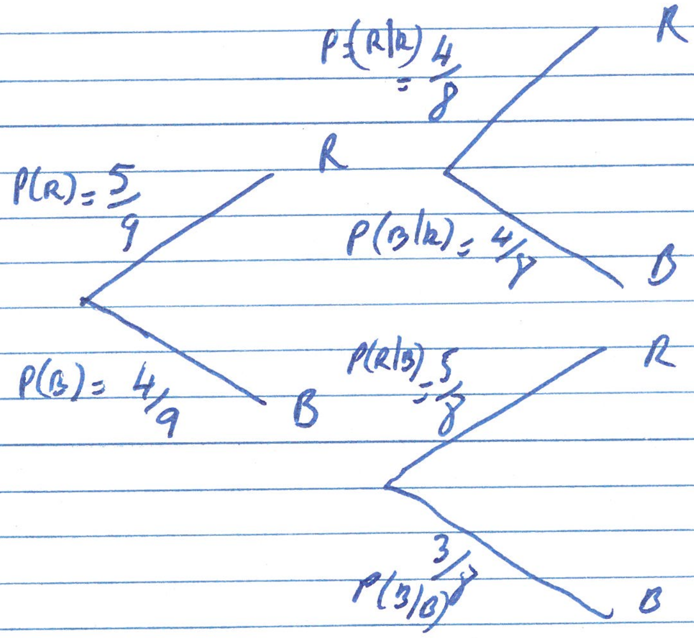
$= 0.45$

(iv) $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.03}{0.45} = \dfrac{1}{15}$

(v) $P(B|A') = \dfrac{P(B \cap A')}{P(A')} = \dfrac{0.3 \times 0.9}{1 - 0.45} = 0.491$

Ex 5E

(2)



(a) $P(R|B) = \frac{5}{8}$

(b) $P(B|R) = \frac{1}{2}$

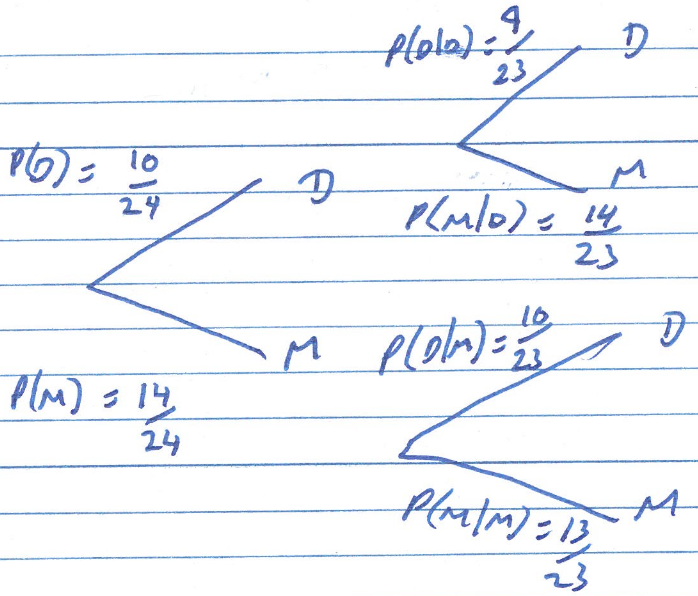
(c) $P(B \cap B) = \frac{4}{9} \times \frac{3}{8} = \frac{12}{72} = \frac{1}{6}$

(d) $P(\text{one red one blue}) = P(R \cap B) + P(B \cap R)$

$$= \left(\frac{5}{9} \times \frac{4}{8}\right) + \left(\frac{4}{9} \times \frac{5}{8}\right)$$

$$= \frac{40}{72} = \frac{5}{9}$$

3)



(a) $P(D \cap D) = \frac{10}{24} \times \frac{9}{23} = \frac{90}{552}$

(b) $P(\text{one dark, one milk}) = P(D \cap M) + P(M \cap D)$

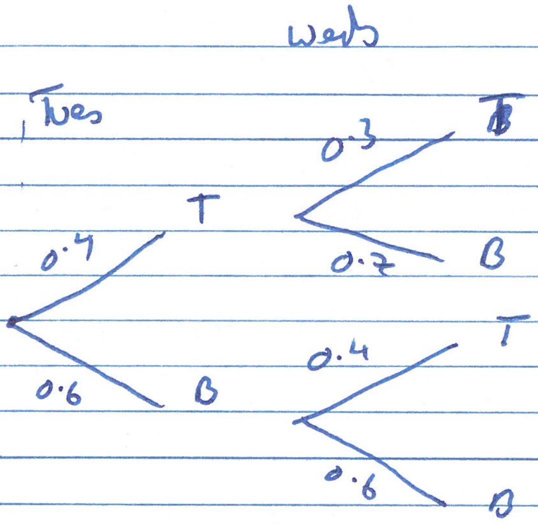
$= \left(\frac{10}{24} \times \frac{14}{23}\right) + \left(\frac{14}{24} \times \frac{10}{23}\right)$

$= \frac{280}{552}$

$= 0.507$

4)

4

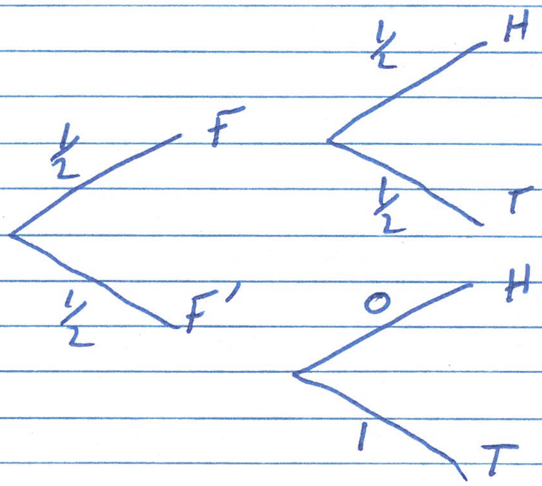


$P(\text{Taxi on Weds}) = (0.4 \times 0.3) + (0.6 \times 0.4)$

$= 0.12 + 0.24$

$= 0.36$

5



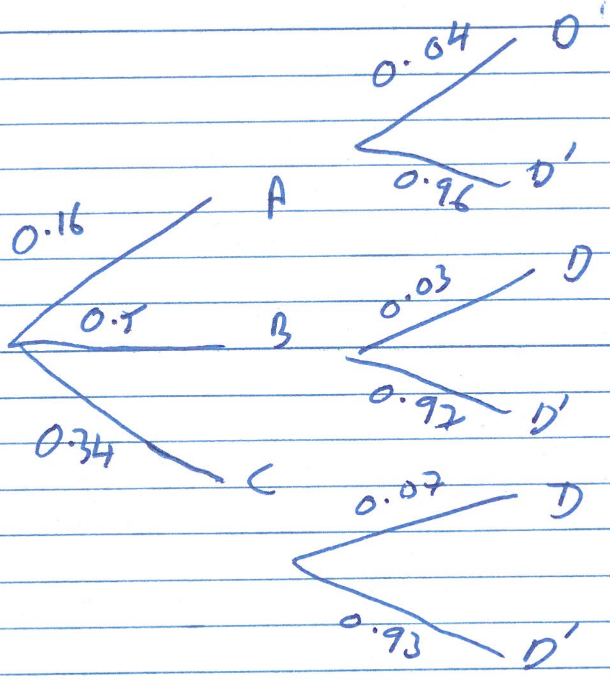
(a) $P(\text{Head}) = P(F \cap H) + P(F' \cap H)$

$= \frac{1}{2} \times \frac{1}{2} + 0$

$= \frac{1}{4}$

(b) $P(F \mid T) = \frac{P(F \cap T)}{P(T)} = \frac{\frac{1}{2} \times \frac{1}{2}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

6



(a) $P(B \cap D) = 0.5 \times 0.03 = 0.015$

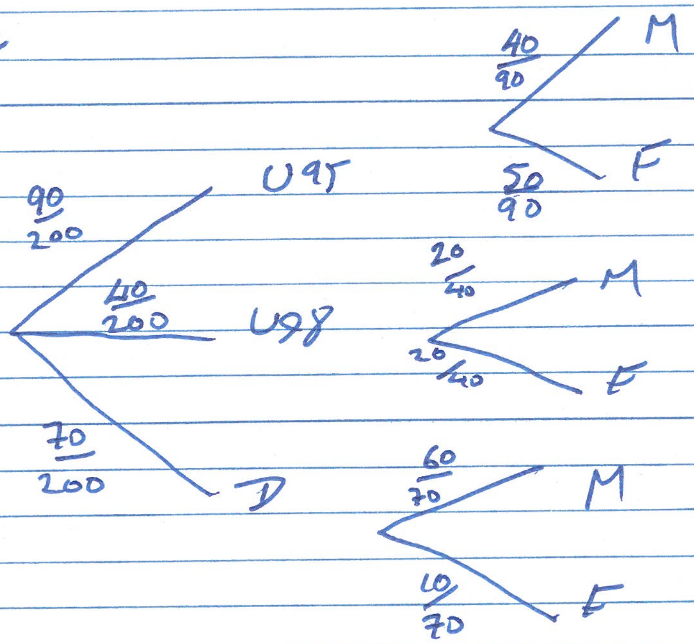
(c) $P(D) = (0.16 \times 0.04) + (0.5 \times 0.03) + (0.34 \times 0.07)$

$= 0.0452$

(d) $P(B|D) = \dfrac{P(B \cap D)}{P(D)} = \dfrac{0.015}{0.0452} = 0.332$

5(a) 8

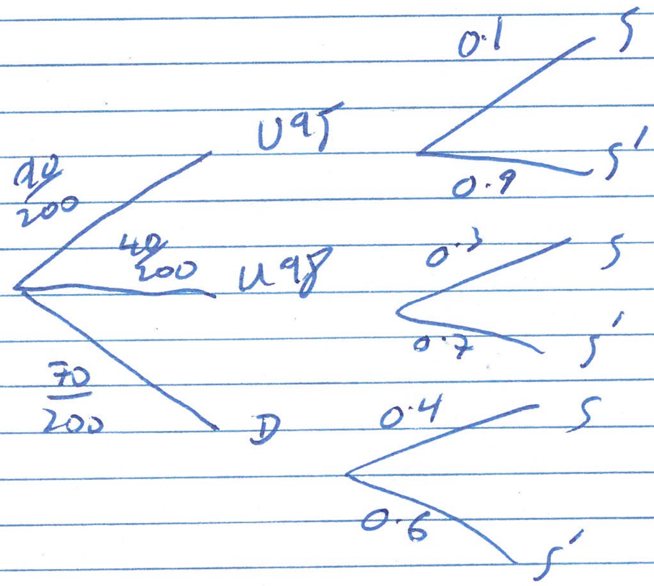
7.



(a) $P(U98) = \frac{40}{200} = 0.2$

(b) $P(M|U98) = \frac{P(M \cap U98)}{P(U98)} = \cancel{\frac{\frac{20}{40} \times \frac{40}{200}}{\frac{120}{200}}} = \frac{\frac{20}{40} \times \frac{40}{200}}{\frac{40}{200}}$

$= 0.5$

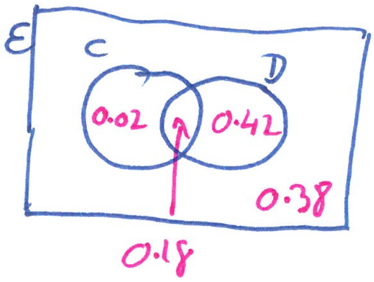


(c) $P(S) = \left(\frac{90}{200} \times 0.1\right) + \left(\frac{40}{200} \times 0.3\right) + \left(\frac{70}{200} \times 0.4\right) = 0.245$

(d) $P(D|S) = \frac{P(D \cap S)}{P(S)} = \frac{\frac{70}{200} \times 0.4}{0.245} = \frac{4}{7} = 0.571$

Conditional Probability & Venn Diagrams

Eg7 C and D are two events such that P(C) = 0.2, P(D) = 0.6 and P(C|D) = 0.3. Find
(a) P(D|C)
(b) P(C' ∩ D')
(c) P(C' ∩ D)



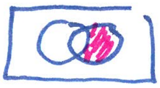
$$P(C|D) = \frac{P(C\cap D)}{P(D)}$$

$$P(C\cap D) = P(C|D)P(D)$$
$$= 0.3 \times 0.6$$
$$= 0.18$$

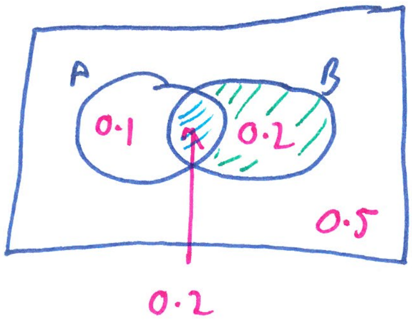
(a) $P(D|C) = \frac{P(D\cap C)}{P(C)} = \frac{0.18}{0.2} = 0.9$

(b) $P(C'\cap D') = 0.38$

(c) $P(C'\cap D) = 0.42$



Eg8 Let A and B be events such that P(A) = 0.3, P(B) = 0.4 and P(A ∪ B) = 0.5. Find
(a) P(B|A)
(b) P(B|A')
(c) P(A)P(B|A) + P(A')P(B|A')
Comment on your answer to (c)



$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$
$$P(A\cap B) = 0.3 + 0.4 - 0.5$$
$$P(A\cap B) = 0.2$$

(a) $P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{0.2}{0.3} = \frac{2}{3}$

(b) $P(B|A') = \frac{P(B\cap A')}{P(A')} = \frac{0.2}{0.7} = \frac{2}{7}$

(c) $\cancel{P(A)} \times \frac{P(A\cap B)}{\cancel{P(A)}} + \cancel{P(A')} \times \frac{P(B\cap A')}{\cancel{P(A')}}$

$P(A\cap B) + P(B\cap A')$

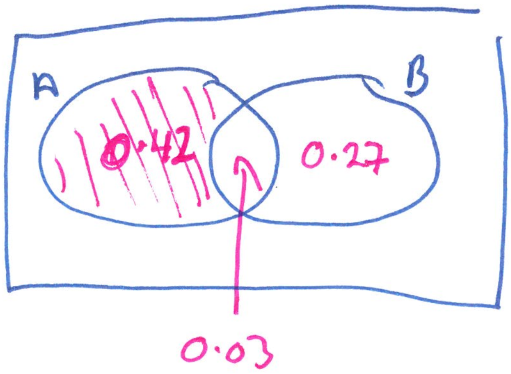
$0.2 + 0.2 = 0.4 = P(B)$ so $P(A)P(B|A) + P(A')(P(B|A') = P(B)$

Eg9 A and B are two events such that P(A|B) = 0.1, P(A|B') = 0.6 and P(B) = 0.3. Find
(a) P(A ∩ B)
(b) P(A ∩ B')
(c) P(A)
(d) P(B|A)
(e) P(B|A')

(a) $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

$0.1 = \dfrac{P(A\cap B)}{0.3}$

$P(A\cap B) = 0.03$



(b) $P(A|B') = \dfrac{P(A\cap B')}{P(B')}$

$0.6 = \dfrac{P(A\cap B')}{0.7}$

$\therefore P(A\cap B') = 0.6\times 0.7 = 0.42$

(c) $P(A) = 0.45$

(d) $P(B|A) = \dfrac{P(B\cap A)}{P(A)} = \dfrac{0.03}{0.45} = \dfrac{1}{15}$

(e) $P(B|A') = \dfrac{P(B\cap A')}{P(A')} = \dfrac{0.27}{0.55} = 0.491$

Exercise 2

1. A and B are two events such that $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \cap B) = 0.4$, find

a $P(A \cup B)$, **b** $P(B \mid A)$, **c** $P(A \mid B)$, **d** $P(A \mid B')$.

2. A and B are two events such that $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \mid B) = 0.4$, find

a $P(B \mid A)$, **b** $P(A' \cap B')$, **c** $P(A' \cap B)$.

3. Let A and B be events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cup B) = \frac{3}{5}$.

Find

a $P(A \mid B)$, **b** $P(A' \cap B)$, **c** $P(A' \cap B')$.

4. C and D are two events and $P(C \mid D) = \frac{1}{3}$, $P(C \mid D') = \frac{1}{5}$ and $P(D) = \frac{1}{4}$, find

a $P(C \cap D)$, **b** $P(C \cap D')$, **c** $P(C)$,
d $P(D \mid C)$, **e** $P(D' \mid C)$, **f** $P(C')$.

5. The events A and B are such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{1}{2}$.

a Show that A and B are independent.

b Represent these probabilities in a Venn diagram.

c Find $P(A \mid B')$.

6. An online readers' club has 50 members. Glasses are worn by 15 members, 18 are left handed and 21 are female. There are four females who are left handed, three females who wear glasses and five members who wear glasses and are left handed. Only one member wears glasses, is left handed and female.

a Draw a Venn diagram to represent these data.

A member is selected at random. Find the probability that the member

b is female, does not wear glasses and is not left handed,

c is male, does not wear glasses and is not left handed,

d wears glasses given that she is left handed and female.

7. For the events J and K,

$P(J \cup K) = 0.5, P(J' \cap K) = 0.2, P(J \cap K') = 0.25.$

a Draw a Venn diagram to represent the events J and K and the sample space S.

Find

b $P(J)$,

c $P(K)$,

d $P(J \mid K)$.

8. A survey of a group of students revealed that 85% have a mobile phone, 60% have an MP3 player and 5% have neither phone nor MP3 player.

a Find the proportion of students who have both gadgets.

b Draw a Venn diagram to represent this information.

Given that a randomly selected student has a phone or an MP3 player,

c find the probability that the student has a mobile phone.

9. A study was made of a group of 150 children to determine which of three cartoons they watch on television. The following results were obtained:

- 35 watch Toontime
- 54 watch Porky
- 62 watch Skellingtons
- 9 watch Toontime and Porky
- 14 watch Porky and Skellingtons
- 12 watch Toontime and Skellingtons
- 4 watch Toontime, Porky and Skellingtons

a Draw a Venn diagram to represent these data.

Find the probability that a randomly selected child from the study watches

b none of the three cartoons,

c no more than one of the cartoons.

A child selected at random from the study only watches one of the cartoons.

d Find the probability that it was Skellingtons.

Two different children are selected at random from the study.

e Find the probability that they both watch Skellingtons.

Numerical Answers

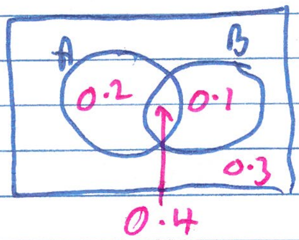
(1a) 0.7 (b) 2/3 (c) 0.8 (d) 0.4 (2a) 0.5 (b) 0.3 (c) 0.3 (3a) 0.3 (b) 0.35 (c) 0.4

(4a) 1/12 (b) 3/20 (c) 7/30 (d) 5/14 (5c) 1/3 (6b) 0.3 (c) 0.14 (d) 0.25

(7b) 0.3 (c) 0.25 (d) 0.2 (8a) 0.5 (c) 17/19 (9b) 0.2 (c) 0.82 (d) 40/93 (e) 0.169

Ex2

(1) $P(A) = 0.6 \quad P(B) = 0.5 \quad P(A \cap B) = 0.4$



(a) $P(A \cup B) = 0.7$

(b) $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.4}{0.6} = \dfrac{2}{3}$

(c) $P(A|B) = \dfrac{0.4}{0.5} = 0.8$

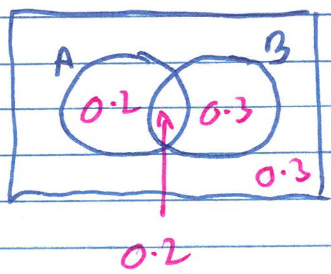
(d) $P(A|B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{0.2}{0.5} = 0.4$

(2) $P(A) = 0.4 \quad P(B) = 0.5 \quad P(A|B) = 0.4$

to find $(A \cap B)$: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$0.4 = \frac{P(A \cap B)}{0.5}$$

$$P(A \cap B) = 0.4 \times 0.5 = 0.2$$



(a) $\cancel{P(A|B) = \dfrac{0.2}{0.5} = 0.4}$

(a) $P(B|A) = \dfrac{0.2}{0.4} = \dfrac{1}{2}$

(b) $P(A' \cap B') = 0.3$

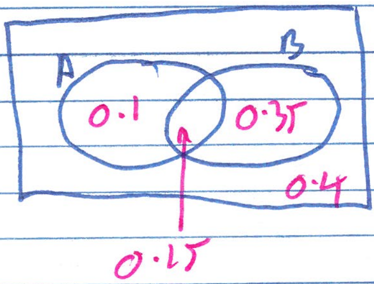
(c) $P(A' \cap B) = 0.3$

(3) $P(A) = \frac{1}{4}$ $P(B) = \frac{1}{2}$ $P(A \cup B) = \frac{3}{5}$

$0.6 = 0.25 + 0.5 - P(A \cap B)$

$P(A \cap B) = 0.15$

(a) $P(A|B) = \frac{0.15}{0.5} = 0.3$



(b) $P(A' \cap B) = 0.35$

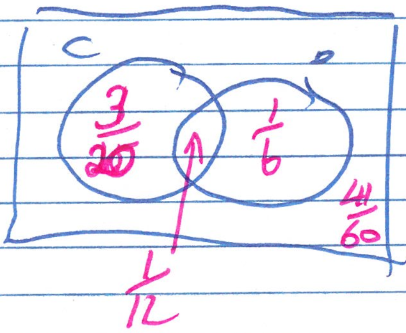
(c) $P(A' \cap B') = 0.4$

(4) $P(C|D) = \frac{1}{3}$ $P(C|D') = \frac{1}{5}$ $P(D) = \frac{1}{4}$

(a) $P(C|D) = \frac{P(C \cap D)}{P(D)}$

$\frac{1}{3} = \frac{P(C \cap D)}{\frac{1}{4}}$

$P(C \cap D) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$



(b) $P(C|D') = \frac{P(C \cap D')}{P(D')}$

$\frac{1}{5} = \frac{P(C \cap D')}{\frac{3}{4}}$

$P(C \cap D') = \frac{3}{20}$

(c) $P(C) = \frac{3}{20} + \frac{1}{12} = \cancel{\frac{7}{30}}\ \frac{7}{30}$

(d) $P(D|C) = \frac{\frac{1}{12}}{\frac{7}{30}} = \frac{5}{14}$

(5) $P(A) = \frac{1}{3}$ $P(B) = \frac{1}{4}$ $P(A \cup B) = \frac{1}{2}$

(a) If independent then $P(A \cup B) = P(A) \times P(B) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

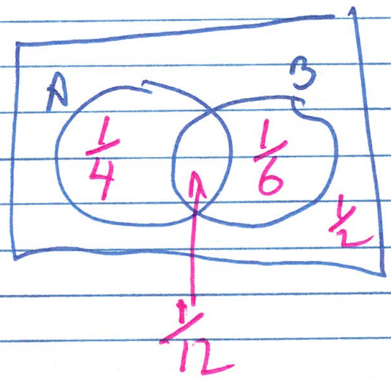
also $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\frac{1}{2} = \frac{1}{3} + \frac{1}{4} - P(A \cap B)$

$P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}$

Consistent ∴ independent.

(b)



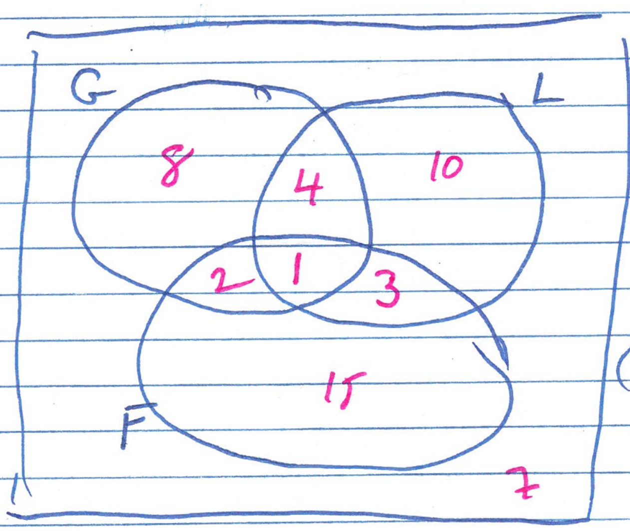
(c) $P(A|B') = \frac{P(A \cap B')}{P(B')}$

$= \frac{1/4}{3/4}$

$= \frac{1}{3}$

(6)

(a)



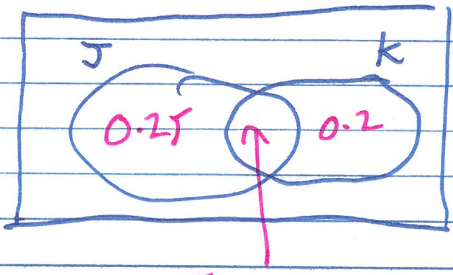
(b) $P(F \cap G' \cap L') = \frac{15}{50} = 0.3$

(c) $P(F' \cap G' \cap L') = \frac{7}{50} = 0.14$

(d) $P(G | L \cap F) = \frac{1}{4} = 0.25$

(7) $P(J \cup K) = 0.5 \quad P(J' \cap K) = 0.2 \quad P(J \cap K') = 0.25$

(a)



$0.5 - 0.25 - 0.2$
$= 0.05$

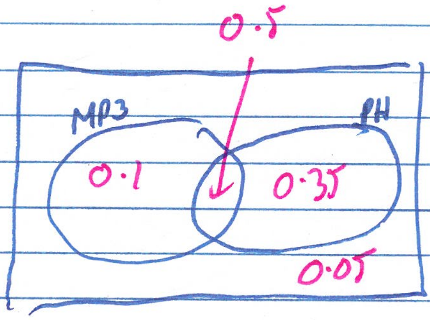
(b) $P(J) = 0.3$

(c) $P(K) = 0.25$

(d) $P(J|K) = \dfrac{0.05}{0.25} = 0.2$

(8)

(b)



(a) $P(MP3 \cup PH) = 0.95$

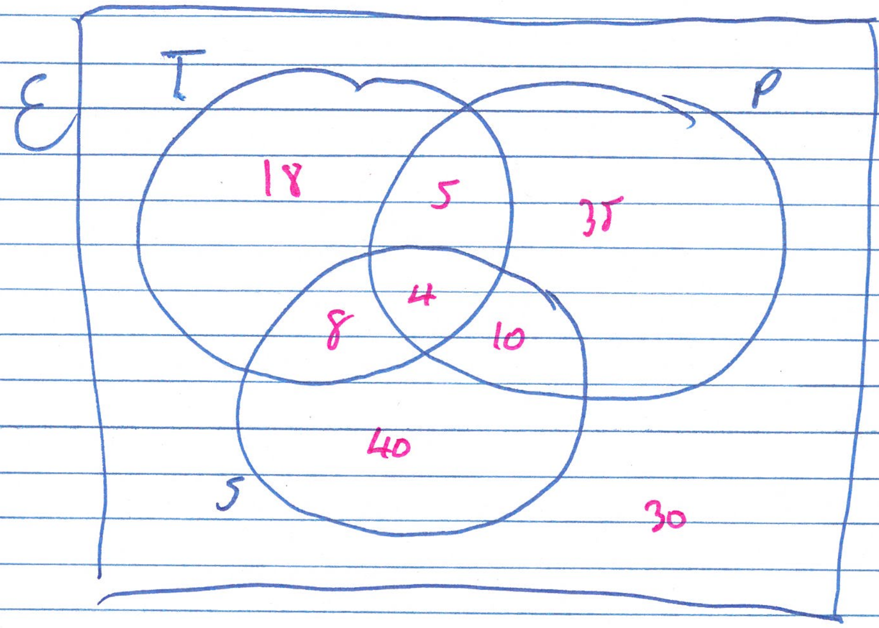
$P(MP3 \cup PH) = P(MP3) + P(PH) - P(MP3 \cap PH)$

$0.95 = 0.6 + 0.85 - P(MP3 \cap PH)$

$P(MP3 \cap PH) = 0.6 + 0.85 - 0.95 = 0.5$

(c) $P(PH | PH \cup MP3) = \dfrac{0.85}{0.95} = 0.895$

9) (a)



(b) $P(\text{none}) = \frac{30}{150} = 0.2$

(c) $P(\text{no more than one}) = \frac{18+35+40+30}{150} = \frac{123}{150} = 0.82$

(d) $P(S \mid \text{only one}) = \frac{P(S \cap \text{only one})}{P(\text{only one})} = \frac{40}{93}$

(e) $P(\text{both watch } S) = \frac{62}{150} \times \frac{61}{149} = 0.169$