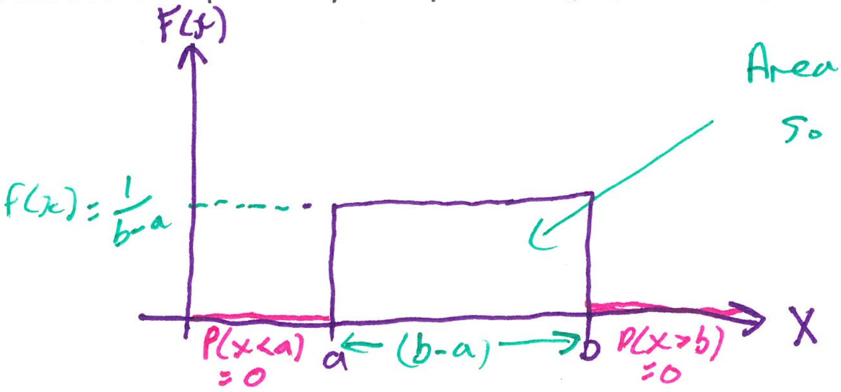
A2 – Continuous Distributions

The continuous uniform (rectangular) distribution X ~ U[a, b]

This has a constant probability density function (pdf) over a range of values and zero elsewhere.



Area of rectangle = 1.

So $f(x) \times (b-a) = 1$

$f(x) = \frac{1}{b-a}$

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

$$\text{Mean, } E(X) = \frac{(a+b)}{2}$$

$$\text{Var}(X) = \frac{1}{12}(b-a)^2$$

Eg5 The continuous variable X is uniformly distributed X ~ U[2, 5]

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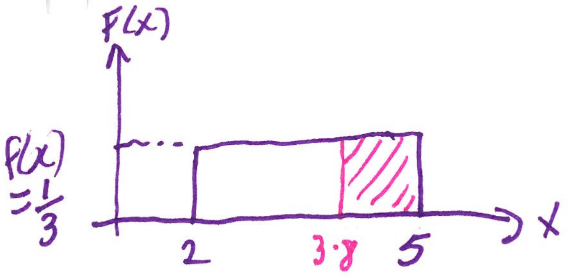
Find (a) E(X) (b) Var(X) (c) P(X > 3.8)



(a) $E(X) = \frac{2+5}{2} = 3.5$

(b) $\text{Var}(X) = \frac{1}{12}(5-2)^2 = \frac{9}{12} = 0.75$

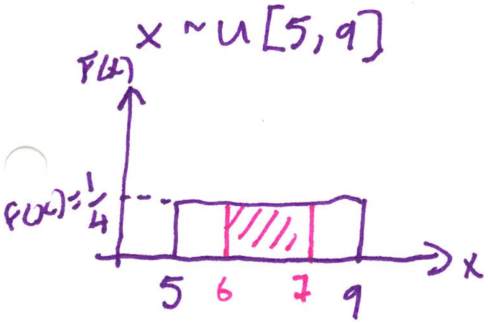
(c) $P(X > 3.8) = (5 - 3.8) \times \frac{1}{3} = 0.4$

Eg6 A junior gymnastics league is open to children who are at least 5 years old but have not yet had their 9^{th} birthdays. The age X years, of a member is modelled as a uniform continuous distribution over the range of possible values between five and nine. Age is measured in years and decimal parts of a year, rather than just completed years. Find

youtu.be/hmTasqdLLn4

(a) the pdf f(x) of X
(b) P(6 ≤ X ≤ 7)
(c) E(X)
(d) Var(X)
(e) The percentage of the children whose ages are within one standard deviation of the mean.

$X \sim U[5,9]$



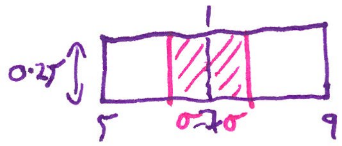
(a) $f(x) = \begin{cases} \frac{1}{4}, & 5 \leq x < 9 \\ 0, & \text{elsewhere} \end{cases}$

(b) $P(6 \leq X \leq 7) = 1 \times \frac{1}{4} = \frac{1}{4}$

(c) $E(X) = \frac{5+9}{2} = 7$

(d) $Var(X) = \frac{1}{12}(9-5)^2 = \frac{16}{12} = \frac{4}{3}$

(e) $\sigma = \sqrt{Var(X)} = \sqrt{\frac{4}{3}}$



% within one σ of mean $= \frac{2 \times \sigma \times 0.25}{1} \times 100 = 57.7\%$

Eg7 A piece of string of length 8cm is randomly cut into two pieces. Find the probability that the longer of the two pieces of string is at least 5cm long.

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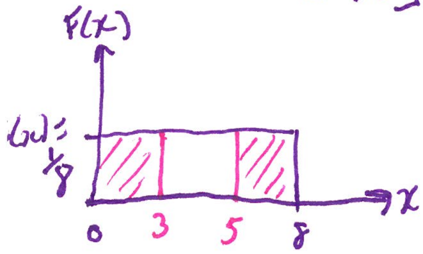
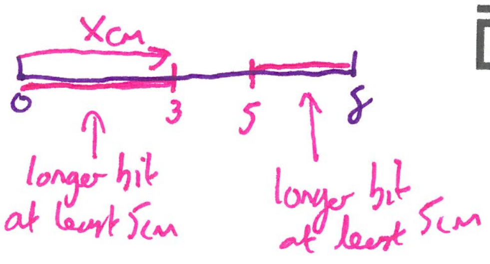
Let X be the distance of the cut from one end.

$\therefore X \sim U[0,8]$





So $P(X \leq 3) + P(X \geq 5)$

$= 1 - (5-3) \times \frac{1}{8}$

$= 1 - \frac{2}{8}$

$= \frac{3}{4}$

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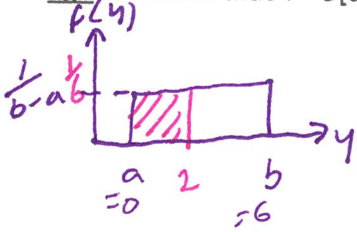
Eg8 Given that Y ~U[a, b] and E(Y) = 3 and Var(Y) = 3, find P(Y < 2).



$E(y)=3 \therefore \frac{a+b}{2}=3 \Rightarrow a+b=6$ — ①

$Var(y)=3 \quad \frac{1}{12}(b-a)^2=3$

$(b-a)^2=36$

$b-a=\pm 6$, but $b>a \therefore b-a=6$ — ②

① + ② $2b=12$

$b=6$

in ① $a=0$

$\therefore Y \sim U[0,6]$

So $P(Y<2)=2 \times \frac{1}{6}=\frac{1}{3}$

Eg9 The amount of time, in minutes, that a person must wait for a bus is represented by the pdf T ~ U[0, 15].

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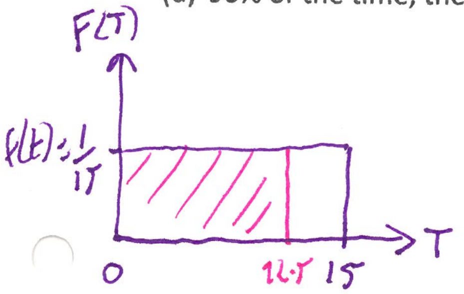
(a) what is the probability that the person waits fewer than 12.5 minutes?
(b) On average, how long must a person wait.
(c) What is the standard deviation of the waiting time?
(d) 90% of the time, the time a person must wait falls below what value?



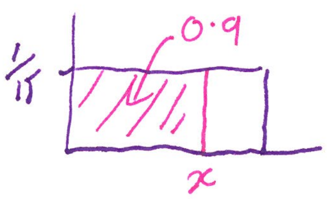
(a) $P(T<12.5)=12.5 \times \frac{1}{15}=0.83$

(b) $E(X)=\frac{0+15}{2}=7.5$ minutes

(c) $\sigma=\sqrt{Var(X)}$, $Var(X)=\frac{1}{12}(15-0)^2=18.75$

$\therefore \sigma=\sqrt{18.75}=4.33$ minutes

(d) $P(T<x)=0.9$



$\frac{x}{15}=0.9$

$x=13.5$

ie 90% of the time a person has to wait no longer than 13.5 mins

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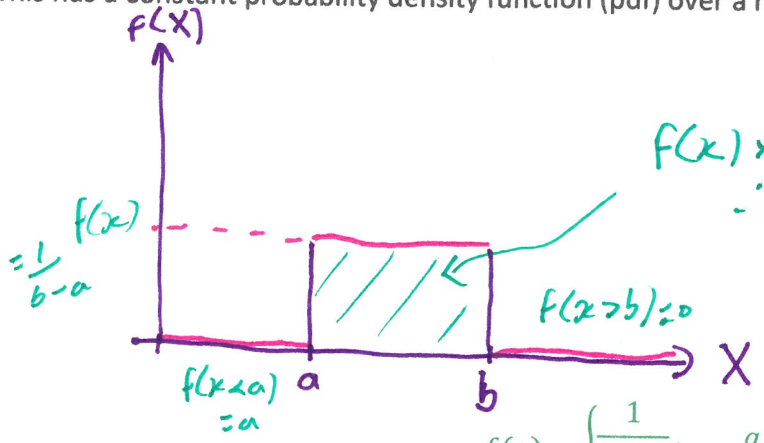

$f(x)\times(b-a)=1$
$\therefore f(x)=\frac{1}{b-a}$

$$f(x)=\begin{cases}\frac{1}{b-a}, & a\le x\le b\\ 0, & elsewhere\end{cases}$$

$$Mean, E(X)=\frac{(a+b)}{2}$$

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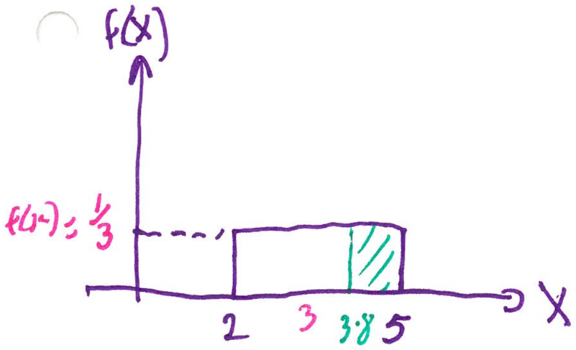

(a) $E(X)=\frac{2+5}{2}=\frac{7}{2}=3.5$

(b) $Var(X)=\frac{1}{12}(5-2)^2=\frac{9}{12}=\frac{3}{4}$

(c) $P(X>3.8)=1-(3.8-2)\times\frac{1}{3}=1-0.6=0.4$

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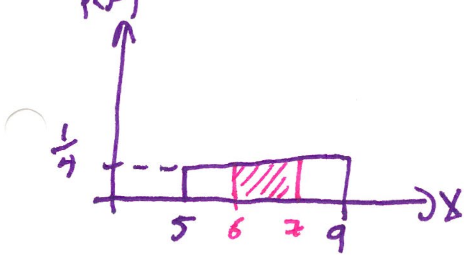
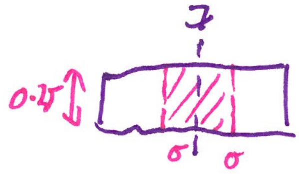


(a) $f(x) = \begin{cases} \frac{1}{4} & \text{for } 5 \le x < 9 \\ 0 & \text{elsewhere} \end{cases}$

(b) $P(6 \le X \le 7) = 1 - \left(1 \times \frac{1}{4}\right) - \left(2 \times \frac{1}{4}\right) = 1 - \frac{3}{4} = \frac{1}{4}$

(c) $E(X) = \frac{5+9}{2} = 7$

(d) $Var(X) = \frac{1}{12}(9-5)^2 = \frac{16}{12} = \frac{4}{3}$



(e) $\sigma = \sqrt{Var(X)} = \sqrt{\frac{4}{3}}$

$\frac{2 \times \sigma \times \frac{1}{4}}{1} \times 100 = 57.7\%$

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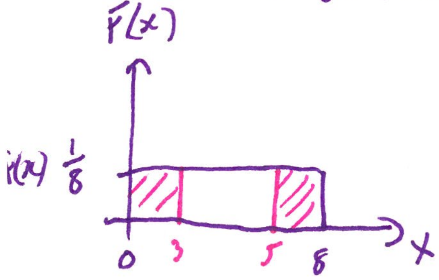


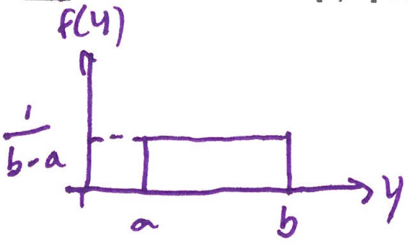
$P(X \le 3) + P(X \ge 5)$

$= 1 - 2 \times \frac{1}{8}$

$= 1 - \frac{1}{4}$

$= \frac{3}{4}$

Eg Given that Y ~U[a, b] and E(Y) = 3 and Var(Y) = 3, find P(Y < 2).

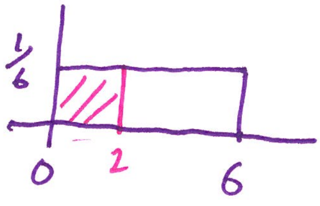


$E(Y) = 3 \therefore \frac{a+b}{2} = 3 \Rightarrow a+b = 6$ — ①

$Var(Y) = 3 \quad \frac{1}{12}(b-a)^2 = 3$

$(b-a)^2 = 36$

$b-a = \pm 6$



If $b - a = +6$ — ②

①+② $2b = 12$

$b = 6$

so $a = 0$

or if $b - a = -6$ — ③

①+③ $2b = 0$

$b = 0$ but $b > a$

So $a = 6$ $\therefore Y \sim U[0, 6]$

$P(Y<2) = 2 \times \frac{1}{6}$

$= \frac{1}{3}$

Eg The amount of time, in minutes, that a person must wait for a bus is represented by the pdf T ~ U[0, 15].

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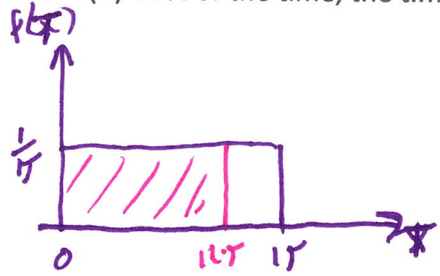
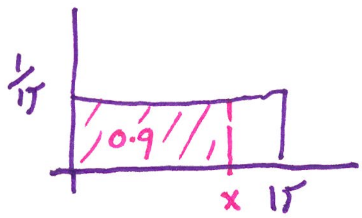


(a) $P(X < 12.5) = 12.5 \times \frac{1}{15} = \frac{5}{6}$

(b) $E(X) = \frac{15}{2} = 7.5$ minutes

(c) $Var(X) = \frac{1}{12}(15)^2 = 18.75$

$\sigma = \sqrt{18.75} = 4.33$ minutes.



$P(T < x) = 0.9$

$\frac{x}{15} = 0.9$

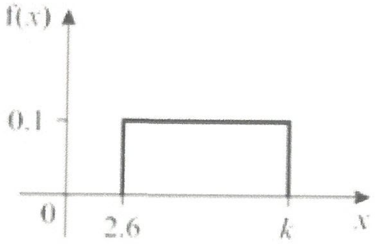
$x = 13.5$

ie 90% of the time a person has to wait no longer than 13.5 mins

Exercise 2.2

1. The continuous random variable $X \sim \mathrm{U}[2, 7]$.
Find
a $\mathrm{P}(3 < X < 5)$,
b $\mathrm{P}(X > 4)$.

2. The continuous random variable X has p.d.f. as shown in the diagram.



Find
a the value of k,
b $\mathrm{P}(4 < X < 7.9)$.

3. The continuous random variable X has p.d.f.

$$\mathrm{f}(x) = \begin{cases} k, & -2 \le x \le 6, \\ 0, & \text{otherwise.} \end{cases}$$

Find
a the value of k,
b $\mathrm{P}(-1.3 < X < 4.2)$.

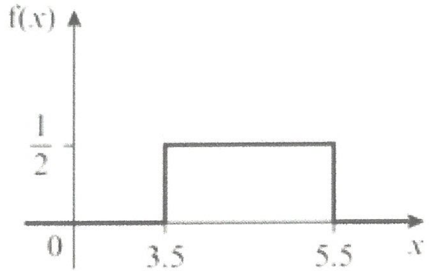
4. The continuous random variable $Y \sim \mathrm{U}[a, b]$. Given that $\mathrm{P}(Y < 5) = \frac{1}{4}$ and $\mathrm{P}(Y > 7) = \frac{1}{2}$, find the value of a and the value of b.

5. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ for the following probability density functions.

a $$\mathrm{f}(x) = \begin{cases} \frac{1}{4}, & 1 \le x \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

b $$\mathrm{f}(x) = \begin{cases} \frac{1}{8}, & -2 \le x \le 6, \\ 0, & \text{otherwise.} \end{cases}$$

6. The continuous random variable X has p.d.f as shown in the diagram.



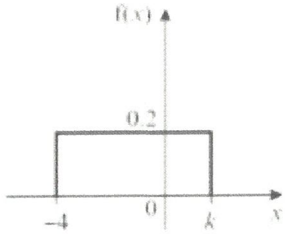
Find:

a $E(X)$,

b $Var(X)$,

7. The continuous random variable $Y \sim U[a, b]$. Given $E(Y) = 1$ and $Var(Y) = \frac{4}{3}$, find the value of a and the value of b.

8. The continuous random variable X has p.d.f as shown in the diagram.



Find

a the value of k,

b $P(-2 < X < -1)$,

c $E(X)$,

d $Var(X)$,

9. A plumber measures, to the nearest cm, the lengths of pipes.

a Suggest a suitable model to represent the difference between the true lengths and the measured lengths.

b Find the probability that for a randomly chosen rod the measured length will be within 0.2 cm of the true length.

c Three pipes are selected at random. Find the probability that all three pipes will be within 0.2 cm of the true length.

Numerical Answers

(1a) 0.4 (b) 0.6 (2a) 12.6 (b) 0.39 (3a) 1/8 (b) 0.6875 (4) a = 3, b = 11

(5a) 3, 4/3 (b) 2, 16/3 (6a) 4.5 (b) 1/3 (7) a = -1, b = 3

(8a) k = 1 (b) 0.2 (c) -1.5 (d) 25/12 (9b) 0.4 (c) 0.064