Differential Equations

A differential equation is an equation connecting x, y and the differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc.

For example: $y^2\frac{dy}{dx} + 2xy = x$ or $\frac{d^2y}{dx^2} - 3y\frac{dy}{dx} + 4y = \sin x$

The order of a differential equation is the order of the differential coefficient of highest order in the equation. So the first example above is of first order, the other of second order.

Different approaches are required to solve these equations depending on their complexity. In Unit 3 we only need concern ourselves with first order differential equations whose variables are separable:

Eg1 Find, for $y > 0$, the general solution of the differential equation $\frac{dy}{dx} = xy$

Notice that when we solve a differential equation, we get not just one solution, but a whole family of solutions, as the constant *A* can take any value. This is called the ***general solution*** of the differential equation. If we are given some more information , we can find out which of the possible conditions is the one that matches the situation in the question.

Eg2 Find the particular solution of the differential equation, $\frac{dy}{dx} = y^2 sinx$ for which $y = 1$ when $x = 0$.

Often differential equations are derived from a scientific context involving rates of change – Newton's Law of Cooling in Physics, radioactive decay in Chemistry, population growth of bacteria of rabbit colonies in Biology would all be modelled using differential equations.

Eg3 The acceleration of an object is inversely proportional to its velocity at any given time and the direction of motion is taken to be positive. When the velocity is $1ms^{-1}$, the acceleration is $3ms^{-2}$.

(a) Find a differential equation to model this situation.
(b) Find the particular solution to this differential equation for which the initial velocity is $2ms^{-2}$.
(c) In this case, how long does the object take to reach a velocity of $8ms^{-1}$?

Eg4 A cold liquid at temperature $\theta°C$, where $\theta < 20$, is standing in a warm room. The temperature of the liquid obeys the differential equation

$$\frac{d\theta}{dt} = 2(20 - \theta)$$

where the time *t* is measured in seconds.

(a) Find the general solution of this differential equation.
(b) Find the particular solution for which $\theta = 5$ when $t = 0$.
(c) In this case, how long does the liquid take to reach a temperature of $18°C$?

Eg 1 $\frac{dy}{dx} = xy$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{x^2}{2} + c$$

$$y = e^{\frac{1}{2}x^2 + c}$$

$$y = e^{\frac{1}{2}x^2} e^{c}$$ Const A

$$y = Ae^{\frac{1}{2}x^2}$$

Eg 2 $\frac{dy}{dx} = y^2 \sin x$

$$\int \frac{1}{y^2}\,dy = \int \sin x\,dx$$

$$\int y^{-2}\,dy = \int \sin x\,dx$$

$$-y^{-1} = -\cos x + c$$

$$-\frac{1}{y} = -\cos x + c$$

when $x=0$, $y=1$

$$-\frac{1}{1} = -1 + c$$

$$c = 0$$

$$\therefore -\frac{1}{y} = -\cos x \Rightarrow y = \sec x$$

Eg 3 $\quad accel \propto \frac{1}{vel}$

(a) but $a = \frac{dv}{dt}$

$$\therefore \frac{dv}{dt} \propto \frac{1}{v}$$

$$\frac{dv}{dt} = \frac{k}{v}$$

when $v=1$, $\frac{dv}{dt} = 3$

$$3 = \frac{k}{1} \Rightarrow k=3$$

$$\therefore \frac{dv}{dt} = \frac{3}{v}$$

(b) $$\int v\,dv = \int 3\,dt$$

$$\frac{v^2}{2} = 3t + c$$

Now @ $t=0$, $v=2$ $\Rightarrow c = 2$

$$\therefore \frac{v^2}{2} = 3t+2$$

$$v^2 = 6t+4$$

$v = \pm\sqrt{6t+4}$, but told direct of motion is +ve

$$\therefore v = \sqrt{6t+4}$$

(c) when $v=8$ $\quad 64 = 6t+4$

$t = 10$ sec

Eg4 $\frac{d\theta}{dt} = 2(20-\theta)$

1a) $\int \frac{1}{20-\theta}\,d\theta = \int 2\,dt$ — (1)

$\Downarrow$

let $u = 20-\theta$

$\frac{du}{d\theta} = -1$

$d\theta = -du$

$\int \frac{1}{u} \cdot -du = -\ln u = -\ln|20-\theta|$

So (1) becomes $-\ln|20-\theta| = 2t + c$

$\ln|20-\theta| = -2t + c$

$20-\theta = e^{-2t+c}$

$20-\theta = Ae^{-2t}$

$\theta = 20 - Ae^{-2t}$

(b) when $t=0$, $\theta=5$ $5 = 20 - A$

$A = 15$

$\therefore \theta = 20 - 15e^{-2t}$

(c) $\theta = 18$ $18 = 20 - 15e^{-2t}$

$15e^{-2t} = 2$

$e^{-2t} = \frac{2}{15} \Rightarrow -2t = \ln\left(\frac{2}{15}\right) \Rightarrow t = -\frac{1}{2}\ln\left(\frac{2}{15}\right)$

$= 1.01$ sec

Eg 5
3.

$$\frac{dP}{dt} = kP$$

(a) $$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln P = kt + c$$

$$P = e^{kt+c}$$

$$P = e^{kt} . e^{c}$$

$$P = Ae^{kt}$$

when $t = 0$, $P = P_0$ $\therefore$ $A = P_0$ $\{ P_0 = Ae^0$, $P_0 = A)$

hence $P = P_0 e^{kt}$ As required.

(b) for $P = 2P_0$ & $k = 2.5$

$$2\cancel{P_0} = \cancel{P_0} e^{2.5t}$$

$$e^{2.5t} = 2$$

take ln's $2.5t = \ln 2$

$$t = \frac{1}{2.5}\ln 2 = 0.277 \text{ days}$$

$\times 24 \times 60 = 399$ minutes

(c) $$\frac{dP}{dt} = \lambda P \cos \lambda t$$

$$\int \frac{1}{P}\,dP = \lambda \int \cos \lambda t\,dt$$

$$\ln P = \frac{\lambda}{\lambda} \sin \lambda t + c$$

$$\ln P = \sin \lambda t + c$$

$$P = e^{\sin \lambda t} . e^{c}$$

egs contd $P = Ae^{\sin\lambda t}$

when $t=0$ $P=P_0 \Rightarrow A = P_0$ $\{P_0 = Ae^{\sin 0} = Ae^0 = A\}$

$\therefore$ $P = P_0 e^{\sin\lambda t}$

(d) now for $P = 2P_0$ & $\lambda = 2.5$

$2\cancel{P_0} = \cancel{P_0} e^{\sin 2.5t}$

$2 = e^{\sin 2.5t}$

take ln's $\ln 2 = \sin 2.5t$

$2.5t = \sin^{-1}(\ln 2)$ $\Leftarrow$ Make sure your calc is in radians!

$t = 0.3063\ldots$ days

$\times 24 \times 60 = 441$ minutes.

Eg5 A population growth is modelled by the differential equation

$$\frac{dP}{dt} = kP,$$

where P is the population, t is the time measured in days and k is a positive constant.

Given that the initial population is P_0,

(a) solve the differential equation, giving P in terms of P_0, k and t. (4)

Given also that $k = 2.5$,

(b) find the time taken, to the nearest minute, for the population to reach $2P_0$. (3)

In an improved model the differential equation is given as

$$\frac{dP}{dt} = \lambda P \cos \lambda t,$$

where P is the population, t is the time measured in days and λ is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,

(c) solve the second differential equation, giving P in terms of P_0, λ and t. (4)

Given also that $\lambda = 2.5$,

(d) find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model. (3)

Exercise 8.1 (WJEC PURE MATHS PPQs)

1. The value, £V, of a car may be modelled as a continuous variable. At time t years, the rate of decrease of V is directly proportional to V^2.

   (a) Write down a differential equation satisfied by V. [1]

   (b) Given that $V = 12000$ when $t = 0$, show that

   $$V = \frac{12000}{at + 1},$$

   where a is a constant. [4]

   (c) The value of the car at the end of two years is £9000. Find the value of the car at the end of four years. [4]

2. The size N of the population of a small island may be modelled as a continuous variable. At time t, the rate of increase of N is directly proportional to the value of N.

(a) Write down the differential equation that is satisfied by N. [1]

(b) Show that $N = Ae^{kt}$, where A and k are constants. [3]

(c) Given that $N = 100$ when $t = 2$ and that $N = 160$ when $t = 12$,

(i) show that $k = 0{\cdot}047$, correct to three decimal places,

(ii) find the size of the population when $t = 20$. [7]

3. Water is leaking from a hole at the bottom of a large tank. The volume of the water in the tank at time t hours is $V\,\text{m}^3$. The rate of decrease of V is directly proportional to V^3.

(a) Write down a differential equation satisfied by V. [1]

(b) Given that $V = 60$ when $t = 0$, show that

$$V^2 = \frac{3600}{at+1},$$

where a is a constant. [4]

(c) When $t = 2$, the volume of the water in the tank is $50\,\text{m}^3$. Find the value of t when the volume of the water in the tank is $27\,\text{m}^3$. Give your answer correct to one decimal place. [4]

4. Part of the surface of a small lake is covered by green algae. The area of the lake covered by the algae at time t years is $A\,\text{m}^2$. The rate of increase of A is directly proportional to $\sqrt{A}$.

(a) Write down a differential equation satisfied by A. [1]

(b) The area of the lake covered by the algae at time $t = 3$ is $64\,\text{m}^2$ and the area covered at time $t = 5{\cdot}5$ is $196\,\text{m}^2$. Find an expression for A in terms of t. [6]

Ex8.1

① rate is decrease $\therefore$ -ve

(a) $\frac{dV}{dt} \propto -V^2$

$\frac{dV}{dt} = -kV^2$

(b) $\int \frac{1}{V^2} dV = \int -k \, dt$

$\int V^{-2} dV = \int -k \, dt$

$\frac{V^{-1}}{-1} = -kt + c$

$-\frac{1}{V} = -kt + c$

when $t=0$, $V=12000$

$-\frac{1}{12000} = c$

$\therefore \quad -\frac{1}{V} = -kt - \frac{1}{12000}$

$-\frac{1}{V} = \frac{-12000kt - 1}{12000}$

$\frac{1}{V} = \frac{1 + 12000kt}{12000}$

let $a = 12000k$

$V = \frac{12000}{at+1}$ as required.

① (c) when $t = 2$, $V = 9000$

$$9000 = \frac{12000}{2a+1}$$

$$2a+1 = \frac{12000}{9000}$$

$$2a+1 = \frac{4}{3}$$

$$2a = \frac{1}{3}$$

$$a = \frac{1}{6}$$

$$\therefore \quad V = \frac{12000}{\frac{t}{6}+1}$$

when $t = 4$, $V = \dfrac{12000}{\frac{4}{6}+1} = £7200$

② (a) rate is increase, $\therefore$ +ve

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = kN$$

(b) $$\int \frac{1}{N}\, dN = \int k\, dt$$

$$\ln N = kt + c$$

$$N = e^{kt+c}$$

$$\cancel{Ae}\ N = e^{kt} . e^{c}$$

$$N = Ae^{kt} \quad \text{As required}$$

(2)(c) when $t=2$, $N=100$

$$100 = Ae^{2k} \quad —(1)$$

when $t=12$, $N=160$.

$$160 = Ae^{12k} \quad —(2)$$

(2) ÷ (1) $$\frac{160}{100} = \frac{Ae^{12k}}{Ae^{2k}}$$

$$1.6 = e^{10k}$$

take ln's $$\ln 1.6 = 10k$$

$$k = \frac{1}{10}\ln 1.6 = 0.047 \text{ as required.}$$

∴ ~~sub~~ in (1) $$100 = Ae^{2(0.047)}$$

$$A = 91.028.$$

$$\therefore N = 91.028e^{0.047t}$$

when $t=20$

$$N = 233.03\ldots, \underline{233}$$

(3) rate of decrease $\therefore$ -ve

(a) $\frac{dV}{dt} \propto -V^3$

$\frac{dV}{dt} = -kV^3$

(b) $\int \frac{1}{-V^3}\, dV = \int k\, dt$

$\int -V^{-3}\, dV = \int k\, dt$

$\frac{-1}{-2}V^{-2} = kt + c$

$\frac{1}{2V^2} = kt + c$

when $t=0$, $V=60$ $\quad c = \frac{1}{2(60)^2} = \frac{1}{7200}$

$\therefore \frac{1}{2V^2} = kt + \frac{1}{7200}$

$\frac{1}{2V^2} = \frac{7200kt + 1}{7200}$

$\frac{1}{V^2} = \frac{7200kt + 1}{3600}$

let $7200k = a$

$V^2 = \frac{3600}{at+1}$ As required.

(3) (a) when $t=2$, $V=50$

$$50^2 = \frac{3600}{2a+1}$$

$$2a+1 = \frac{3600}{2500}$$

$$2a+1 = \frac{36}{25}$$

$$a = \frac{11}{50}$$

$$\therefore \quad V^2 = \frac{3600}{\frac{11t}{50}+1}$$

when $V = 27$ $$27^2 = \frac{3600}{\frac{11t}{50}+1}$$

$$\frac{11t}{50}+1 = \frac{400}{81}$$

$$\frac{11t}{50} = \frac{319}{81}$$

$t = 17.9$ hours

Q4 (a) $\frac{dA}{dt} \propto \sqrt{A}$

$$\frac{dA}{dt} = kA^{\frac{1}{2}}$$

(b) $\int A^{-\frac{1}{2}} \, dA = \int k \, dt$

$$2A^{\frac{1}{2}} = kt + c$$

when $t = 3$, $A = 64$

$$2(64)^{\frac{1}{2}} = 3k + c$$

$$16 = 3k + c \quad — (1)$$

when $t = 5.5$, $A = 196$

$$2(196)^{\frac{1}{2}} = 5.5k + c$$

$$28 = 5.5k + c \quad — (2)$$

(2) − (1) $12 = 2.5k$

$$k = 4.8$$

in (1) $16 = 3(4.8) + c$

$$c = 1.6$$

$\therefore$ $2A^{\frac{1}{2}} = 4.8t + 1.6$

$$A^{\frac{1}{2}} = 2.4t + 0.8$$

$$A = (2.4t + 0.8)^2$$

Exercise 8.2 (WJEC MECHANICS PPQs)

1. A vehicle P, of mass 800 kg, on a straight horizontal road passes the point O with velocity $5\ \text{ms}^{-1}$. At time t s later its velocity is $v\ \text{ms}^{-1}$ and the vehicle is subject to a resistance given by $(4000 + 1600v)$ N.

   (a) Show that v satisfies the differential equation

$$\frac{dv}{dt} = -(5 + 2v)\ . \qquad [2]$$

   (b) (i) Find the time when P is at rest.

   (ii) Find an expression for v in terms of t. [9]

2. A car of mass 600kg starts from rest and moves along a straight horizontal road. At time t s, the force acting on the car has magnitude $\dfrac{27000}{(t+3)^2}$ N acting in the direction of motion.

   Resistance to motion may be ignored.

   (a) Find an expression for $v\ \text{ms}^{-1}$, the velocity of the car at time t s. Hence show that the speed of the car has a limiting value as t increases and find this limiting value. [7]

   (b) Calculate the distance travelled by the car in the first 6 s of motion. Give your answer correct to two decimal places. [5]

3. At time $t = 0$, a particle of mass 6 kg is projected vertically upwards from a point A with a speed of $24{\cdot}5\ \text{ms}^{-1}$. The resistance acting on the particle has magnitude $3v$ N, where $v\ \text{ms}^{-1}$ is the speed of the particle at time t s.

   (a) (i) Show that v satisfies the equation

$$2\frac{dv}{dt} = -19{\cdot}6 - v.$$

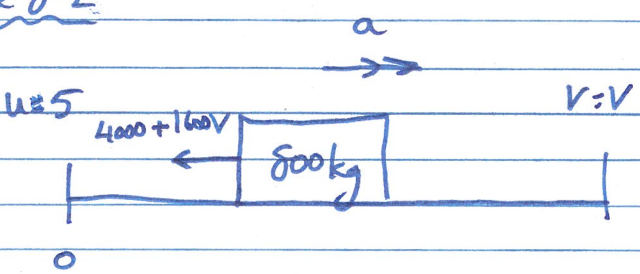
   (ii) Find an expression for v in terms of t. [8]

   (b) Determine the time when the particle reaches its maximum height. [2]

   (c) Find an expression for x in terms of t, where x m is the distance of the particle from A at time t s. [4]

Ex 8.2

1



(a) N2L $\quad -(4000+1600V) = 800a$

but $a = \frac{dv}{dt}$ $\quad \therefore \; -4000 - 1600V = 800\frac{dv}{dt}$

$\div 800$

$$-5 - 2V = \frac{dV}{dt}$$

$$\frac{dv}{dt} = -(5+2V) \quad \text{as required.}$$

(b) $$\int \frac{1}{5+2v}\,dv = -\int dt$$

$$\frac{1}{2}\ln|5+2v| = -t + c$$

when $t=0$, $V=5$

$$c = \frac{1}{2}\ln 15$$

$$\therefore \frac{1}{2}\ln|5+2V| = -t + \frac{1}{2}\ln 15 \qquad -(1)$$

$$t = \frac{1}{2}\ln 15 - \frac{1}{2}\ln|5+2V|$$

when $V=0$ $\quad t = \frac{1}{2}(\ln 15 - \ln 5) = \frac{1}{2}\ln 3 = 0.55$ sec

(1) (b)(ii) From (1) $\frac{1}{2}\ln|5+2v| - \frac{1}{2}\ln 15 = -t$

$$\ln\left|\frac{5+2v}{15}\right| = -2t$$

$$\frac{5+2V}{15} = e^{-2t}$$

$$5+2V = 15e^{-2t}$$

$$2V = 15e^{-2t} - 5$$

$$V = \frac{5}{2}\left[3e^{-2t} - 1\right]$$

(2) (a) N2L $\frac{27000}{(t+3)^2} = 600a$

but $a = \frac{dv}{dt}$

$$\frac{27000}{(t+3)^2} = 600\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{45}{(t+3)^2}$$

$$\int dv = 45\int \frac{1}{(t+3)^2}\,dt$$

$$\int V = 45\int (t+3)^{-2}\,dt$$

$$V = \frac{-45}{t+3} + c$$

when $t=0$, $V=0$, $0 = -\frac{45}{3} + c \Rightarrow c = 15$

② (a) contd $V = 15 - \frac{45}{t+3}$

As $t \Rightarrow \infty$, $V \Rightarrow 15$

(b) ~~Vn~~ $x = \int v \, dt$

$x = \int 15 - \frac{45}{t+3} \, dt$

$t > 0$, so no need for $| \; |$

$x = 15t - 45\ln(t+3) + c$

when $t=0$, $x=0$

$0 = 0 - 45\ln 3 + c \Rightarrow c = 45\ln 3$

$\therefore x = 15t - 45\ln(t+3) + 45\ln 3$

$x = 15t - 45\ln\left(\frac{t+3}{3}\right)$

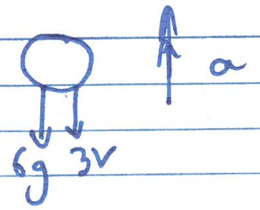
when $t=6$, $x = 15(6) - 45\ln\left(\frac{9}{3}\right)$

$x = 90 - 45\ln 3$

$x = 40.56$ metres

③

(a)



(i) <u>N2L</u> $-6g - 3v = 6a$

but $a = \frac{dv}{dt}$ $\quad -6g - 3v = 6\frac{dv}{dt}$

$\div 3 \quad -2g - v = 2\frac{dv}{dt}$

$\therefore \quad 2\frac{dv}{dt} = -19.6 - v$ as required.

(ii) $2\frac{dv}{dt} = -(19.6 + v)$

$2\int \frac{1}{19.6+v}\,dv = -\int dt$

$2\ln|19.6+v| = -t + C$

when $t=0$, $v = 24.5$

$C = 2\ln 44.1$

So $2\ln|19.6+v| = -t + 2\ln 44.1$

$-t = 2\ln\left|\frac{19.6+v}{44.1}\right|$

$-t/2 = \ln\left|\frac{19.6+v}{44.1}\right|$

(3)(a)(ii) cont

$$\frac{19.6+V}{44.1} = e^{-t/2}$$

$$19.6+V = 44.1e^{-t/2}$$

$$V = 44.1e^{-t/2} - 19.6$$

(b) @ Max ht, $V=0$

$$44.1e^{-t/2} = 19.6$$

$$-\frac{t}{2} = \ln\left(\frac{19.6}{44.1}\right)$$

$$-t = 2\ln\left(\frac{19.6}{44.1}\right)$$

$$t = -2\ln\left(\frac{19.6}{44.1}\right) = 1.62 \text{ sec}$$

(c) $x = \int v \, dt$

$$x = \int 44.1e^{-t/2} - 19.6 \, dt$$

$$x = 44.1 \times \frac{1}{-\frac{1}{2}}e^{-t/2} - 19.6t + c$$

$$x = -88.2e^{-t/2} - 19.6t + c$$

when $t=0$, $x=0$

$$0 = -88.2e^{0} - 0 + c$$

$$c = 88.2$$

$$\therefore x = 88.2 - 88.2e^{-t/2} - 19.6t$$