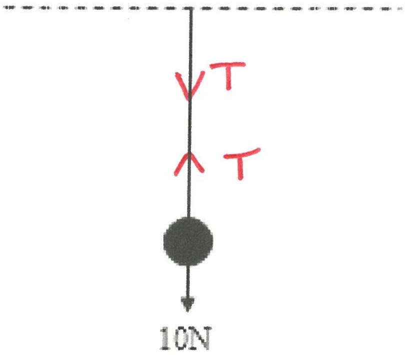
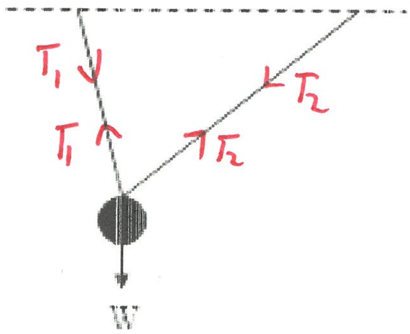
7. More on Equilibrium & Moments

Equilibrium

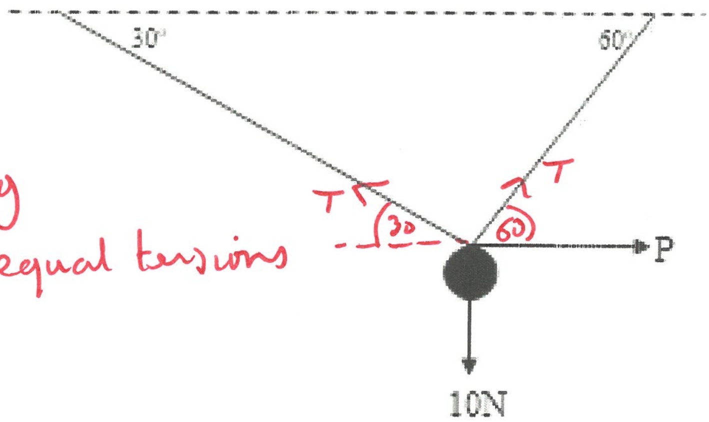
- When the resultant of a system of forces acting on a particle is zero, the particle is said to be in equilibrium.
- A particle which isn't accelerating, but is travelling with a constant velocity is in equilibrium. ie a particle doesn't have to be stationary to be in equilibrium.
- A particle on the point of moving when the frictional force opposing motion reaches its maximum possible magnitude is said to be at limiting equilibrium.
- When a system is in equilibrium, each part of the system is in equilibrium





Eg1 A string is tied to two points on the same level and a ***smooth ring*** of weight 10N which can slide freely along the string is pulled by a horizontal force, *P*. For the position of equilibrium shown in the diagram, find *P* and the tension in the string.

Smooth ring indicates equal tensions either side



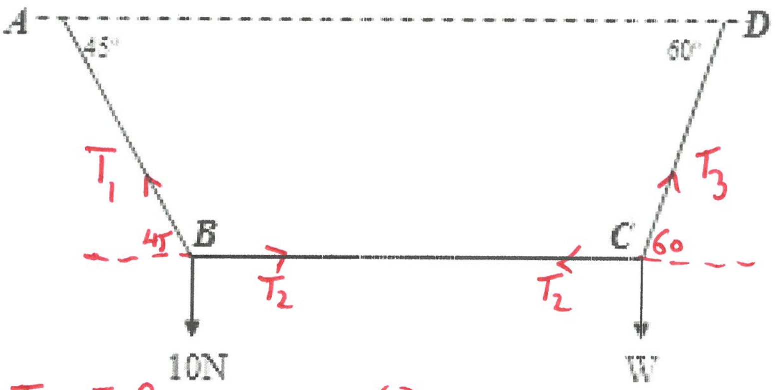
$\Sigma F_x = 0 \qquad P + T\cos 60 - T\cos 30 = 0 \quad — (1)$

$\Sigma F_y = 0 \qquad T\sin 30 + T\sin 60 - 10\cancel{g} = 0 \quad — (2)$

From (2) $T = \dfrac{10}{\sin 30 + \sin 60} = 7.3\ N\ (2sf)$

In (1) $P = 7.3[\cos 30 - \cos 60] = 2.7\ N$

Eg2 ABCD is a string knotted at B and C. Find W and the tensions in AB, BC and CD.



@ B: $\Sigma F_x \quad T_2 - T_1 \cos 45 = 0 \quad —(1)$

$\Sigma F_y \quad T_1 \sin 45 - 10 = 0 \quad —(2)$

@ C: $\Sigma F_x \quad T_3 \cos 60 - T_2 = 0 \quad —(3)$

$\Sigma F_y \quad T_3 \sin 60 - W = 0 \quad —(4)$

from (2) $T_1 = \frac{10}{\sin 45} = 10\sqrt{2}\,N$

in (1) $T_2 = 10\sqrt{2}\cos 45 = 10N$

in (3) $T_3 = \frac{10}{\cos 60} = 20N$

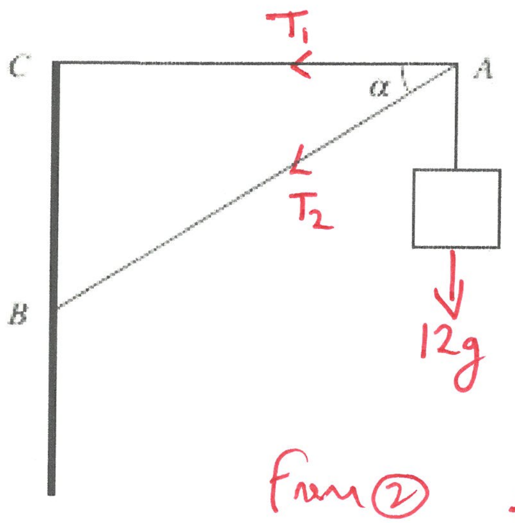
in (4) $W = 20 \sin 60 = 10\sqrt{3}\,N$

Eg3 The diagram shows a sign attached to a point A. It is supported by two light rods AB and AC.

The rod AC is horizontal and the rod AB is at an angle of α to the horizontal, where $\sin \alpha = 0.6$.

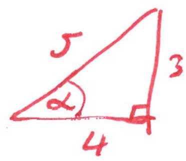
The mass of the sign is 12kg.

Calculate the forces acting along each rod, indicating clearly whether the rods are in thrust or tension.



$\Sigma F_x: \ -T_1 - T_2 \cos\alpha = 0 \quad —(1)$

$\Sigma F_y: \ -T_2 \sin\alpha - 12g = 0 \quad —(2)$



$\sin\alpha = 0.6$

$\cos\alpha = 0.8$

from (2) $+T_2 = \frac{12g}{-0.6} = -196\,N$

in (1) $T_1 = -T_2 \cos\alpha$

$= -(-196)\times 0.8$

$= 156.8$

T_1 is in tension, T_2 is in thrust

xercise 7.1 (WJEC PPs)

1. The diagram shows an object A, of mass 15 kg, suspended in equilibrium in a shaft with vertical walls by means of two ropes XY and XZ. The rope XY makes an angle of 45° with the vertical and the rope XZ makes an angle of 60° with the vertical.

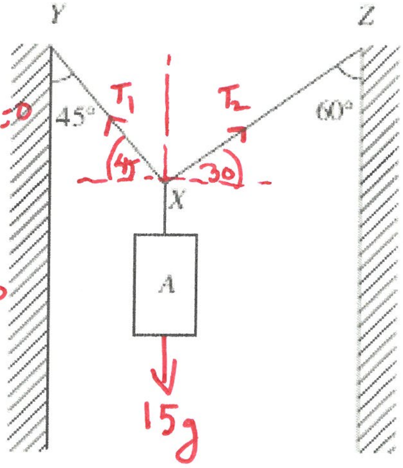
$\Sigma F_x: \quad T_2 \cos 30 - T_1 \cos 45 = 0$

$\Sigma F_y: \quad T_1 \sin 45 + T_2 \sin 30 - 15g = 0$

from ① $T_2 = \dfrac{T_1 \cos 45}{\cos 30}$ — ③

in ② ~~$T_1 \sin 45 = 15g - T_2 \sin 30$~~

~~$T_1 \sin 45 = 15g -$~~



in ② $T_1 \sin 45 + \dfrac{T_1 \cos 45 \sin 30}{\cos 30} = 15g$

$T_1 \left[\sin 45 + \cos 45 \tan 30\right] = 15g$

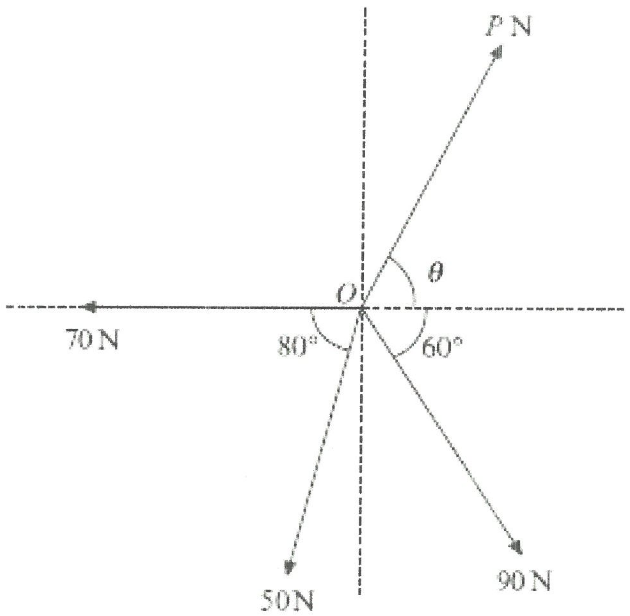
$T_1 = 131.796\ldots = 130\text{ N}$ 2sf.

in ③ $T_2 = 131.796\ldots\left(\dfrac{\cos 45}{\cos 30}\right) = 107.61\ldots$

$= 110\text{ N}$ 2sf

Calculate the tension in each of the ropes XY and XZ. [7]

2. The diagram shows four horizontal forces acting at a point O. The forces are in equilibrium.



Calculate the value of P and the size of the angle θ. Give each of your answers correct to one decimal place. [8]

$\Sigma F_x: \quad P\cos\theta + 90\cos 60 - 50\cos 80 - 70 = 0$ — ①

$\Sigma F_y: \quad P\sin\theta - 90\sin 60 - 50\sin 80 = 0$ — ②

From ① $P\cos\theta = 70 + 50\cos 80 - 90\cos 60$ — ③

From ② $P\sin\theta = 90\sin 60 + 50\sin 80$ — ④

④ ÷ ③ $\tan\theta = 3.7759\ldots$

$\theta = 75.2°$

in ④ $P = \dfrac{90\sin 60 + 50\sin 80}{\sin 75.2} = 131.6\text{ N}.$

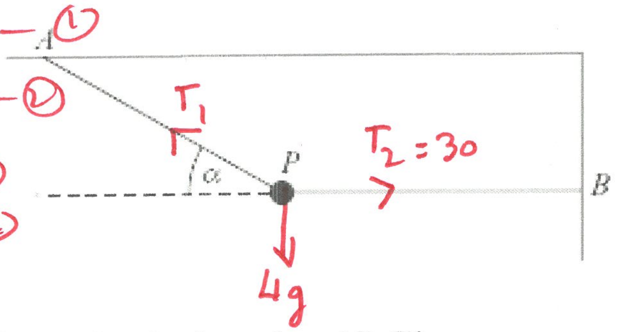
3. The diagram shows a particle P, of mass 4 kg, held in equilibrium by two light inextensible strings AP and BP. The string AP makes an angle α with the horizontal and is attached to the ceiling at the point A. The string BP is horizontal and is attached to the wall at the point B. The tension in the string BP is 30 N.

$\Sigma F_x: 30 - T_1 \cos\alpha = 0$ — (1)

$\Sigma F_y: T_1 \sin\alpha - 4g = 0$ — (2)

From (1) $T_1 \cos\alpha = 30$ — (3)

From (2) $T_1 \sin\alpha = 4g$ — (4)



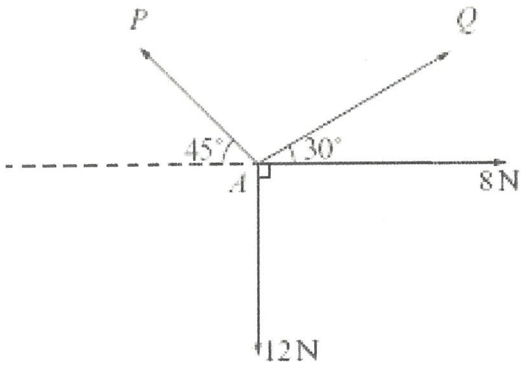
(4) ÷ (3) $\tan\alpha = \frac{4g}{30}$

$\alpha = 52.57°$ ✓

In (1) $T_1 = \frac{30}{\cos 52.57} = 49.3$ N ✓

Find the angle α and the tension in the string AP. Give your answers correct to 2 decimal places. [8]

4. The diagram shows four forces acting at a point A in a horizontal plane.



Given that the forces are in equilibrium, calculate the value of P and the value of Q. Give your answers correct to one decimal place. [7]

$\Sigma F_x: Q\cos 30 + 8 - P\cos 45 = 0$ — (1)

$\Sigma F_y: Q\sin 30 + P\sin 45 - 12 = 0$ — (2)

(1) + (2) $Q\cos 30 + Q\sin 30 - 4 = 0$

$Q = \frac{4}{\cos 30 + \sin 30} = 2.9\text{N}$ ✓

In (1) $P\cos 45 = (2.9)\cos 30 + 8$

$P = 14.9\text{N}$. ✓

The Moment of a Force

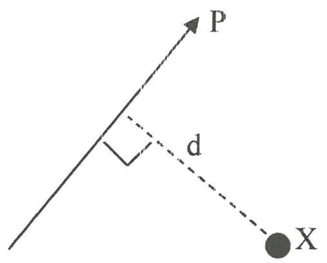
From our everyday experience we know that:

- it is easier to undo a tight nut using a long spanner when the force is applied to the end of the spanner, rather than by using a short spanner.
- If a boy sits at one end of a see-saw which is pivoted at its centre, he can be balanced by a heavier boy sitting nearer to the centre of the see-saw.
- A door is more easily closed by pushing on the edge further from the hinges, rather than by pushing at a point part way across the door.

In each of these examples, the application of the force is causing a body to rotate about an axis, ie rotational motion. Previously, only motion along a line has been considered, ie translational motion.

Definition

The moment of a force about a point is found by multiplying the magnitude of the force by the perpendicular distance from the point to the line of action of the force.



The moment of the force P about the point X is $P \times d$

A force will have no moment about a point on its line of action as d = 0.

If the force is measured in Newtons and the distance in metres, the moment of the force is measured in Newton metres (Nm).

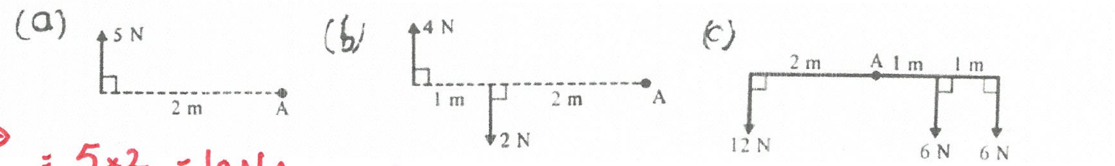
Sense of Rotation

A nut is usually rotated in an anticlockwise direction when being undone. All rotations should have their sense clearly stated (eg, +ve clockwise, -ve anticlockwise): the moment of a force about a point has both magnitude and direction.

Algebraic Sum of Moments

If a number of coplanar forces act on a body, their moments about any point may be added provided due regard is given to the sense of each moment.

Eg4 For each of the situations below, find the total moment about the point A.



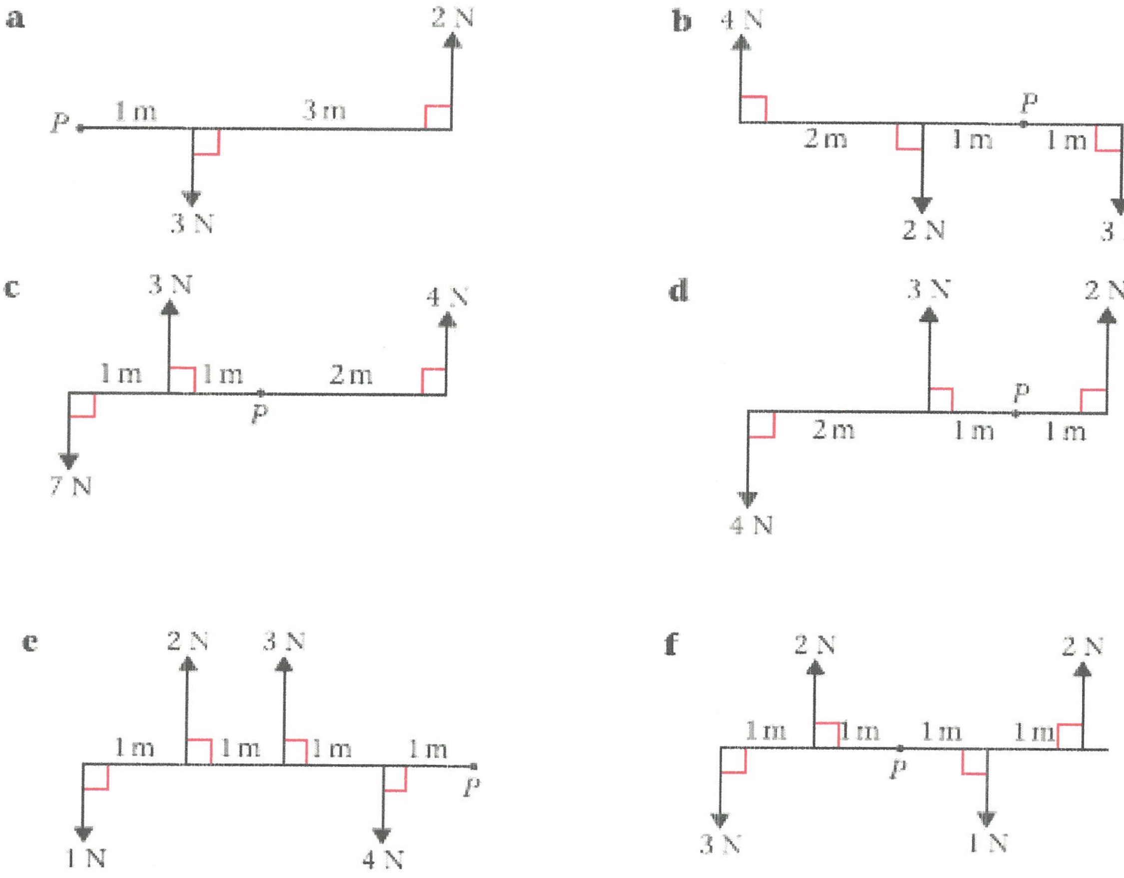
$\Sigma \circlearrowright_A = 5 \times 2 = 10\ Nm$

$\Sigma \circlearrowright_A : (4 \times 3) + (-2 \times 2)$
$= 12 - 4$
$= 8\ Nm$

$\Sigma \circlearrowright_A : (-12 \times 2) + (6 \times 1) + (6 \times 2)$
$= -24 + 6 + 12$
$= -6\ Nm$

Exercise 7.2

1 These diagrams show sets of forces acting on a light rod. For each rod, calculate the sum of the moments about P.



(a) $\Sigma \circlearrowright_P = (3 \times 1) + (-2 \times 4) = 3 - 8 = -5\ Nm$

(b) $\Sigma \circlearrowright_P = (4 \times 3) + (-2 \times 1) + (3 \times 1) = 13\ Nm$

(c) $\Sigma \circlearrowright_P = (-7 \times 2) + (3 \times 1) + (-4 \times 2) = -19\ Nm$
$-14 \quad +3 \quad -8$

(d) $\Sigma \circlearrowright_P = (-4 \times 3) + (3 \times 1) + (-2 \times 1) = -11\ Nm$
$-12 \quad +3 \quad -2$

(e) $\Sigma \circlearrowright_P : (-1 \times 4) + (2 \times 3) + (3 \times 2) + (-4 \times 1) = 4\ Nm$
$-4 \quad +6 \quad +6 \quad -4$

(f) $\Sigma \circlearrowright_P = (-3 \times 2) + (2 \times 1) + (1 \times 1) + (-2 \times 2) = -7\ Nm$
$-6 \quad +2 \quad +1 \quad -4$

Ex 7.2 Answers

1 a 5 Nm anticlockwise
b 13 Nm clockwise
c 19 Nm anticlockwise
d 11 Nm anticlockwise
e 4 Nm clockwise
f 7 Nm anticlockwise

Parallel Forces in Equilibrium

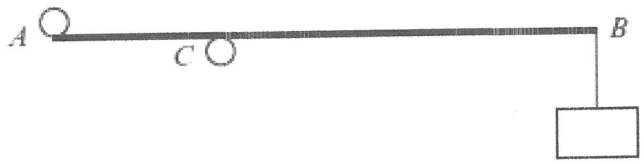
For parallel forces to be in equilibrium, two conditions must hold true:

1. the component of the resultant force in any direction must be zero,
2. the algebraic sum of the moments about any point must be zero, ie the sum of the anti-clockwise moments must equal the sum of the clockwise moments.

Eg5 A uniform beam, of length 2m and mass 4kg, has a mass of 3kg attached at one end and a mass of 1kg attached at the other end. Find the position of the support if the beam rests in a horizontal position.

Eg6 A light horizontal beam of length 2m rests with ends A and B on smooth supports. The beam carries masses of 5kg and 2kg at distances of 60cm and 150cm respectively from A. Find the reaction at each support.

Eg7 The diagram shows a body, of mass 65 kg, attached to the end B of a uniform rigid rod AB of length 4 m. The mass of the rod is 35 kg. The rod is held horizontally in equilibrium by two smooth cylindrical pegs, one at A and another at C, where $AC = 1{\cdot}2$ m.



(a) Write down the moment of the weight of the rod about the point A. State your units clearly. [2]

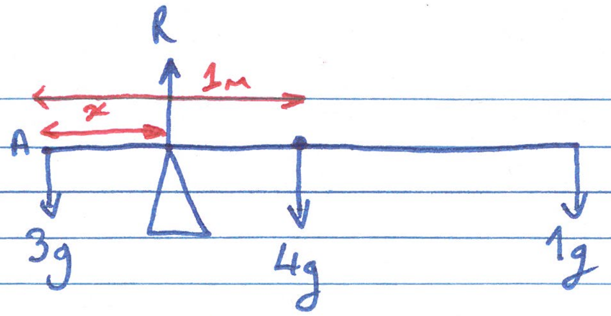
(b) Find the forces exerted on the rod at A and C. [6]

Non-Uniform Rods (rod – assumes no thickness and no bending)

The centre of mass of a non-uniform rod is located at some point other than the midpoint of the rod.

Eg8 A non-uniform rod AB of length 4m and mass 5kg is in equilibrium in a horizontal position resting on two supports at points C and D where AC = 1m and AD = 2m. The magnitude of the reaction at C is half the magnitude of the reaction at D. Find the distance of the centre of mass of the rod from A.

Eg5



in equilibrium $\therefore \Sigma C = 0, \ \Sigma F_y = 0$

$$R - 8g = 0$$

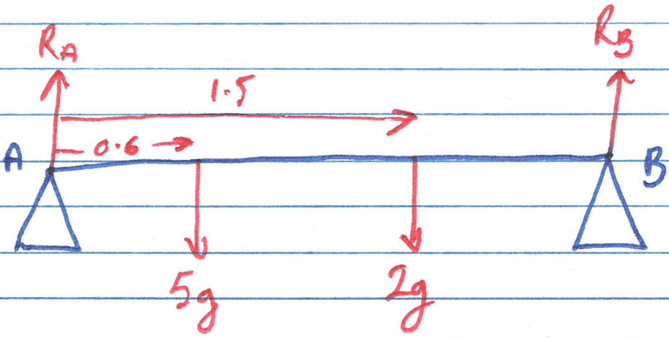
$$R = 8g$$

$$\Sigma C_A: \ (-R \times x) + (4g \times 1) + (1g \times 2) = 0$$

$$8gx = 6g$$

$$x = \frac{6}{8} = 0.75\text{m}$$ from 3kg mass.

Eg6.



light beam so negligible weight.

System in equilibrium $\therefore \Sigma C = 0, \ \Sigma F_y = 0$

$$\Sigma F_y: \ R_A + R_B - 5g - 2g = 0$$

$$R_A + R_B = 7g \quad -(1)$$

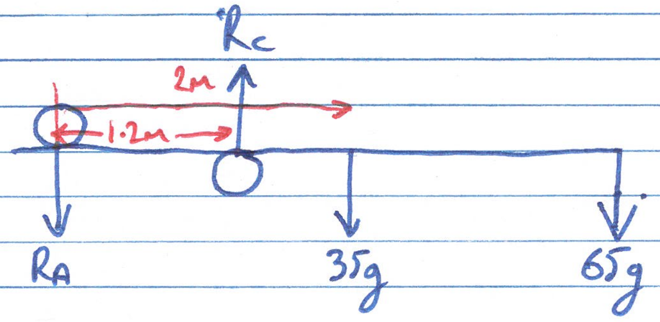
$$\Sigma C_A: \ (5g \times 0.6) + (2g \times 1.5) + (-R_B \times 2) = 0$$

$$2R_B = 6g$$

$$R_B = 3g \text{ N}$$

in (1) $R_A = 7g - 3g = 4g$ N

Eg7



(a) $35g \times 2 = 70g = 686\ Nm$ (690 to 2sf)

(b) $\Sigma F_y = 0 \quad R_C - R_A - 35g - 65g = 0$

$$R_C - R_A = 100g \quad — (1)$$

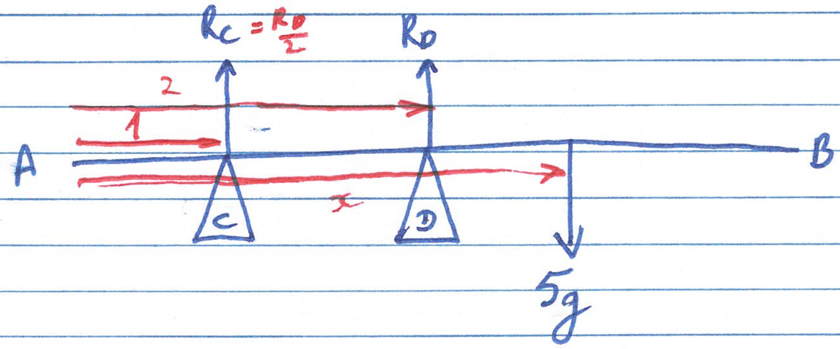
$$\Sigma \circlearrowleft_A = (-R_C \times 1.2) + (70g) + (65g \times 4) = 0$$

$$1.2 R_C = 330g$$

$$R_C = \frac{330g}{1.2} = 2695\ N$$

in (1) $R_A = 2695 - 100g = 1715\ N$

Eg8



$\Sigma F_y = 0 \quad R + \frac{R}{2} - 5g = 0$

$$\frac{3R}{2} = 5g$$

$$R = \frac{10g}{3}$$

Eg. Contd

$$\Sigma \circlearrowright_A : \left(-\frac{R}{2} \times 1\right) + \left(-R \times 2\right) + \left(5g\, x\right) = 0$$

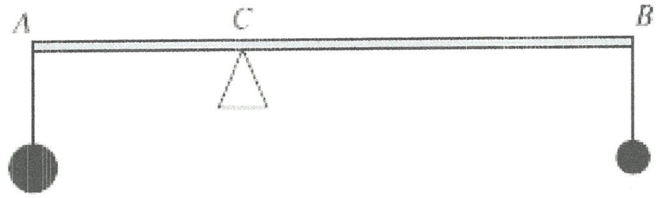
$$\cancel{5} g x = \cancel{5}\frac{R}{2}$$

$$\cancel{g} x = \frac{1}{2}\left(\frac{10\cancel{g}}{3}\right)$$

$$x = \frac{5}{3}\ \text{m} \quad \text{from } A.$$

Exercise 7.3 (WJEC PPs)

1. A uniform rod AB, of mass 3 kg, has length 2 m. A particle of mass 5 kg is attached to the end A, and a particle of mass 2 kg is attached to the end B. The diagram shows the rod resting horizontally in equilibrium on a smooth support at the point C, where $AC = x$ m.



Calculate the magnitude of the reaction of the support at C and the value of x. [6]

2. The diagram shows a uniform rod AB, of mass 4 kg and length 1·6 m, with a particle, of mass 0·5 kg, attached at a point C of the rod, where $AC = 0{\cdot}5$ m.
The rod is resting horizontally in equilibrium on two smooth supports at points X and Y of the rod, where $AX = 0{\cdot}6$ m and $AY = 1{\cdot}2$ m.

(a) Calculate the reaction at X and the reaction at Y. [7]

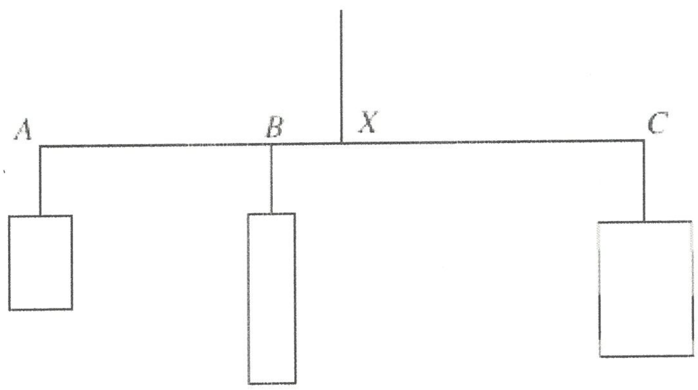
(b) When an additional particle of mass M kg is attached to the point C, the rod is on the point of turning about X. Calculate the value of M. [4]

3. A uniform beam AB, of length 6 m, rests in a horizontal position on two smooth supports at C and D, where $AC = 1$ m and $BD = 1{\cdot}2$ m, as shown in the diagram.

(a) When a vertical force of magnitude 1800 N is applied upwards to the beam at the end A, the beam is about to tilt about the support at D.
Determine the weight of the beam. [5]

(b) The vertical force is now removed so that the beam is resting in equilibrium on the two supports. Calculate the magnitude of the reaction of each of the supports at C and D on the beam. [5]

4. The diagram shows a wind chime consisting of a horizontal uniform rod AC, suspended in equilibrium by means of a light string attached to the mid-point X of the rod, together with three objects hanging from the points A, B and C of the rod.

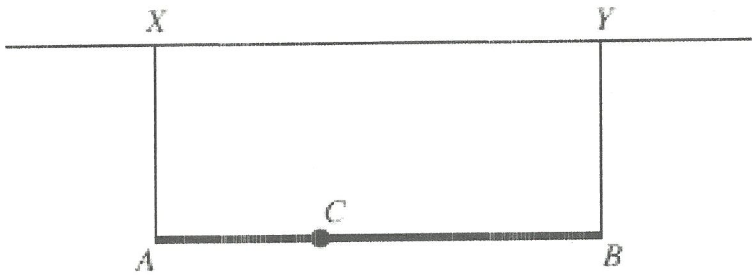


The length of the rod AC is 20 cm and the length of AB is 8 cm. The masses of the objects hanging from A, B, C are 0·1 kg, M kg, 0·4 kg respectively. The mass of the rod is 0·5 kg.

(a) Find the value of M. [4]

(b) Calculate the tension in the string. [3]

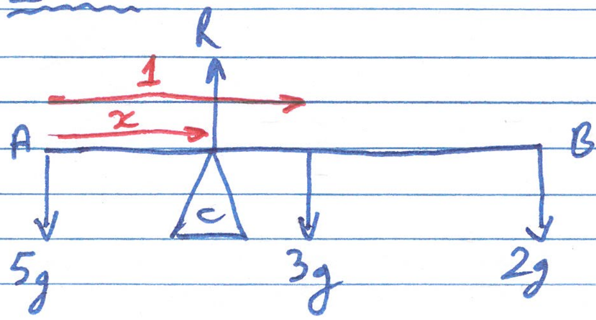
5. A uniform rod AB is suspended horizontally from the ceiling by means of two vertical light inextensible strings XA and YB of equal length.



The rod AB has mass 6 kg and length 1·4 m. A particle, of mass 10 kg, is attached to the rod at point C, where $AC = 0{\cdot}3$ m. Calculate the tension in each of the strings XA and YB. [7]

Ex 7.3

(1)



$\Sigma F_y = R - 5g - 3g - 2g = 0$

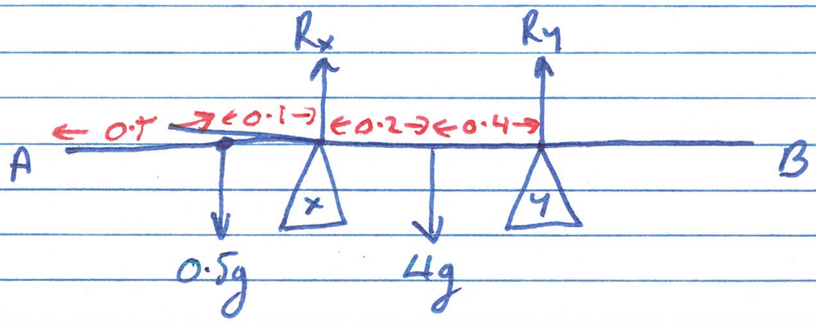
$R = 10g = 98N$ ✓

$\Sigma C_A = (-Rx) + (3g \times 1) + (2g \times 2) = 0$

$98x = 7g$

$x = \frac{7g}{98} = 0.7m$ ✓

(2)



(a) $R_x + R_y - 0.5g - 4g = 0$

$R_x + R_y = 4.5g \quad -(1)$

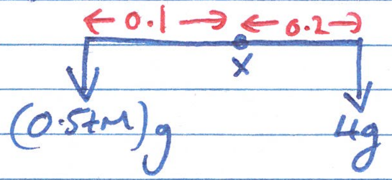
$\Sigma C_x : (-0.5g \times 0.1) + (4g \times 0.2) + (-R_y \times 0.6) = 0$

$0.6R_y = 0.75g$

$R_y = 12.25N$ ✓

in (1) $R_x = 4.5g - 12.25 = 31.85N$ ✓

(2)(b) On point of turning about X, $R_Y = 0$



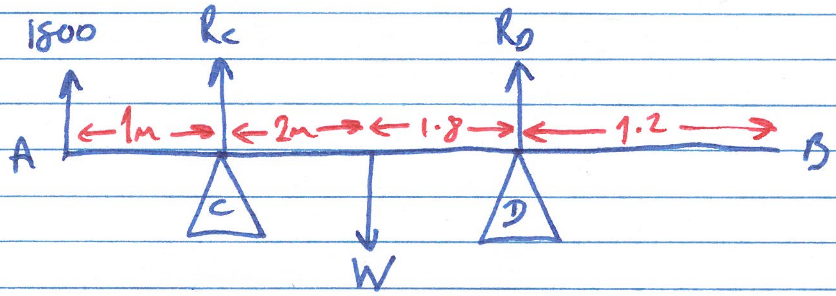
$\Sigma \circlearrowright_X : \; (0.5 + M)g \times 0.1 = 4g \times 0.2$

$0.05 + 0.1M = 0.8$

$0.1M = 0.75$

$M = 7.5\,kg$ ✓

(3)



(a) @ point of turning about D, $R_C = 0$

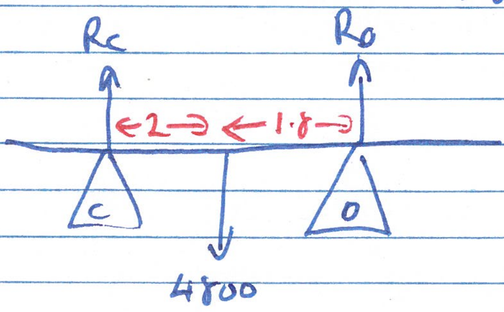
~~$1800 + R_D - W = 0$~~

~~$W = 1800 + R_D$~~

$\Sigma \circlearrowright_D : \; (1800 \times 4.8) + (-W \times 1.8) = 0$

$W = \dfrac{1800 \times 4.8}{1.8} = 4800\,N$ ✓

(b)



$R_C + R_D - 4800 = 0$ — (1)

$\Sigma \circlearrowright_C : \; (4800 \times 2) + (-R_D \times 3.8) = 0$

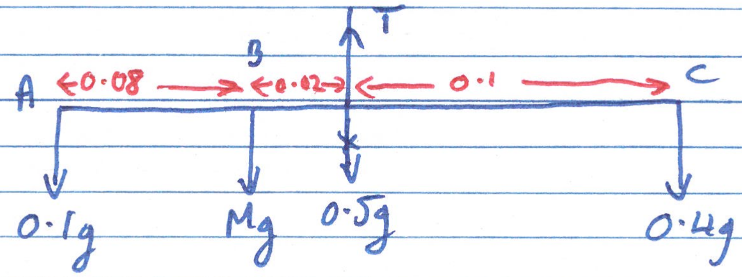
$R_D = \dfrac{4800 \times 2}{3.8} = 2526.31\ldots$ ✓

$= 2500\,N$

in (1) $R_C = 4800 - 2526.31 = 2273.68\ldots$ ✓

$= 2300\,N$

④



(a) $\sum \curvearrowright_x : (-0.1\cancel{g} \times 0.1) + (-M\cancel{g} \times 0.02) + (0.4\cancel{g} \times 0.1) = 0$

$-0.01 - 0.02M + 0.04 = 0$

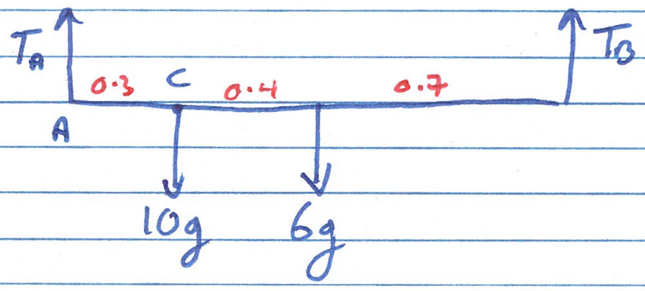
$0.02M = 0.03$

$M = 1.5\,kg$ ✓

(b) $T - 0.1g - 1.5g - 0.5g - 0.4g = 0$

$T = 2.5g = 24.5N$ ✓

⑤



$T_A + T_B = 16g$ — ①

$\sum \curvearrowright_A : (10g \times 0.3) + (6g \times 0.7) + (-T_B \times 1.4) = 0$

$3g + 4.2g = 1.4T_B$

$T_B = \frac{7.2g}{1.4} = 50.4\ N$ ✓

u ① $T_A = 16g - 50.4 = 106.4\ N$. ✓