Unit 3 – 6.Further Differentiation Techniques

Parametric Differentiation

To differentiate a function which is defined in terms of a parameter *t*, you need to use the chain rule:

If $x = f(t)$ and $y = f(t)$

$$\frac{dx}{dt} = \frac{d}{dt}[f(t)] \qquad \frac{dy}{dt} = \frac{d}{dt}[f(t)]$$

We can use chain rule to find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{d\square} \times \frac{d\square}{dx}$$, where $\square$ is the 3rd parameter, t in our case

So $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

And $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$

Hence $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Eg1 A curve has the parametric equations $x = t^2$ and $y = 2t$. Find

(i) $\frac{dy}{dx}$ in terms of the parameter *t*;

(ii) the equation of the tangent to the curve at the point where $t = 3$;

(iii) by eliminating the parameter, find the Cartesian equation of the curve.

Eg2 An ellipse has parametric equations $x = 4\cos\theta$ and $y = 3\sin\theta$. Find

(i) $\frac{dy}{dx}$ at the point with parameter θ;

(ii) the equation of the normal at the point where $\theta = \frac{\pi}{4}$

(iii) by eliminating the parameter, find the Cartesian equation of the curve.

Eg1 $x = t^2$ — (1) $y = 2t$ — (2)

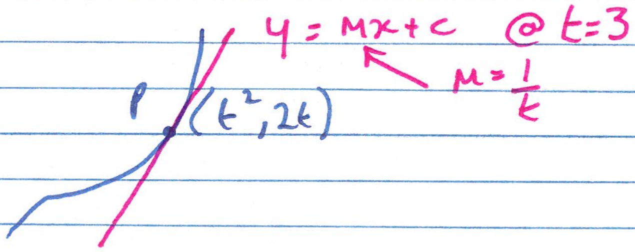
(i) $\frac{dx}{dt} = 2t$ $\frac{dy}{dt} = 2$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2 \times \frac{1}{2t}$$

$$= \frac{1}{t}$$

(ii)



eqⁿ of tangent $y - 2t = \frac{1}{t}(x - t^2)$

@ $t=3$ $y - 6 = \frac{1}{3}(x - 9)$

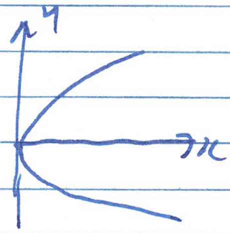
$3y - 18 = x - 9$

$3y - x = 9$

(iii) from (2) $t = \frac{y}{2}$

in (1) $x = \frac{y^2}{4}$

$\therefore$ $y^2 = 4x$ (eqⁿ of a parabola)



Eg2 $x = 4\cos\theta \quad — (1)$ $\qquad y = 3\sin\theta \quad — (2)$

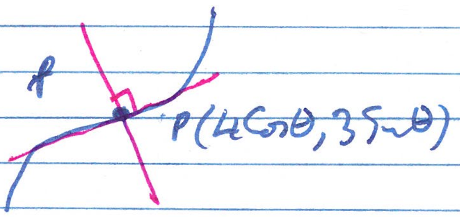
(i) $\dfrac{dx}{d\theta} = -4\sin\theta \qquad \dfrac{dy}{d\theta} = 3\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 3\cos\theta \times \frac{1}{-4\sin\theta}$$

$$\frac{dy}{dx} = -\frac{3}{4}\cot\theta$$

(ii)



$$M_{norm} = -\frac{1}{M} = +\frac{4}{3}\tan\theta$$

eqⁿ of norm $\quad y - 3\sin\theta = \frac{4}{3}\tan\theta\,(x - 4\cos\theta)$

@ $\theta = \frac{\pi}{4}$ $\quad y - 3\sin\frac{\pi}{4} = \frac{4}{3}\tan\frac{\pi}{4}\left(x - 4\cos\frac{\pi}{4}\right)$

$$y - \frac{3}{\sqrt{2}} = \frac{4}{3}\left(x - \frac{4}{\sqrt{2}}\right)$$

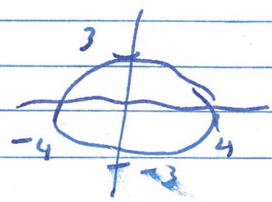
$\times 3\sqrt{2}$ $\quad 3\sqrt{2}y - 9 = 4\sqrt{2}x - 16$

$$3\sqrt{2}y - 4\sqrt{2}x + 7 = 0$$

(iii) From (1) $\cos\theta = \dfrac{x}{4}$ $\qquad$ From (2) $\sin\theta = \dfrac{y}{3}$

Now $\sin^2\theta + \cos^2\theta = 1$

$$\frac{y^2}{16} + \frac{x^2}{9} = 1 \quad \Leftarrow \text{eq}^n \text{ of ellipse}$$



Exercise 3.6A (1st column)

1 Find $\frac{dy}{dx}$ for each of the following, leaving your answer in terms of the parameter t:

a $x = 2t,\ y = t^2 - 3t + 2$ **b** $x = 3t^2,\ y = 2t^3$ **c** $x = t + 3t^2,\ y = 4t$

d $x = t^2 - 2,\ y = 3t^5$ **e** $x = \frac{2}{t},\ y = 3t^2 - 2$ **f** $x = \frac{1}{2t-1},\ y = \frac{t^2}{2t-1}$

g $x = \frac{2t}{1+t^2},\ y = \frac{1-t^2}{1+t^2}$ **h** $x = t^2e^t,\ y = 2t$ **i** $x = 4\sin 3t,\ y = 3\cos 3t$

j $x = 2 + \sin t,\ y = 3 - 4\cos t$ **k** $x = \sec t,\ y = \tan t$ **l** $x = 2t - \sin 2t,\ y = 1 - \cos 2t$

2 **a** Find the equation of the tangent to the curve with parametric equations $x = 3t - 2\sin t,\ y = t^2 + t\cos t$, at the point P, where $t = \frac{\pi}{2}$.

b Find the equation of the tangent to the curve with parametric equations $x = 9 - t^2,\ y = t^2 + 6t$, at the point P, where $t = 2$.

3 **a** Find the equation of the normal to the curve with parametric equations $x = e^t,\ y = e^t + e^{-t}$, at the point P, where $t = 0$.

b Find the equation of the normal to the curve with parametric equations $x = 1 - \cos 2t,\ y = \sin 2t$, at the point P, where $t = \frac{\pi}{6}$.

4 Find the points of zero gradient on the curve with parametric equations

$x = \frac{t}{1-t},\ y = \frac{t^2}{1-t},\ t \neq 1.$

You do not need to establish whether they are maximum or minimum points.

Answers

1 **a** $\frac{2t-3}{2}$ **b** $\frac{6t^2}{6t} = t$ **c** $\frac{4}{1+6t}$

d $\frac{15t^3}{2}$ **e** $-3t^3$ **f** $t(1-t)$

g $\frac{2t}{t^2-1}$ **h** $\frac{2}{(t^2+2t)e^t}$ **i** $-\frac{3}{4}\tan 3t$

j $4\tan t$ **k** $\operatorname{cosec} t$ **l** $\cot t$

2 **a** $y = \frac{\pi}{6}x + \frac{\pi}{3}$ **b** $2y + 5x = 57$

3 **a** $x = 1$ **b** $y + \sqrt{3}x = \sqrt{3}$

4 $(0, 0)$ and $(-2, -4)$

1(g) $x = \frac{2t}{1+t^2}$

$y = uv$

$y = \frac{u}{v}$

OR $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

let $u = 2t$ $\quad v = 1+t^2$

$\frac{du}{dt} = 2$ $\quad \frac{dv}{dt} = 2t$

Now $\frac{dx}{dt} = \frac{(1+t^2)2 - 2t(2t)}{(1+t^2)^2}$

$\frac{dx}{dt} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$

$\frac{dx}{dt} = \frac{2-2t^2}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$

$y = \frac{1-t^2}{1+t^2}$

let $u = 1-t^2$ $\quad V = 1+t^2$

$\frac{du}{dt} = -2t$ $\quad \frac{dV}{dt} = 2t$

Now $\frac{dy}{dt} = \frac{(1+t^2)^-2t - (1-t^2)2t}{(1+t^2)^2}$

$\frac{dy}{dt} = \frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}$

$\frac{dy}{dt} = \frac{-4t}{(1+t^2)^2}$

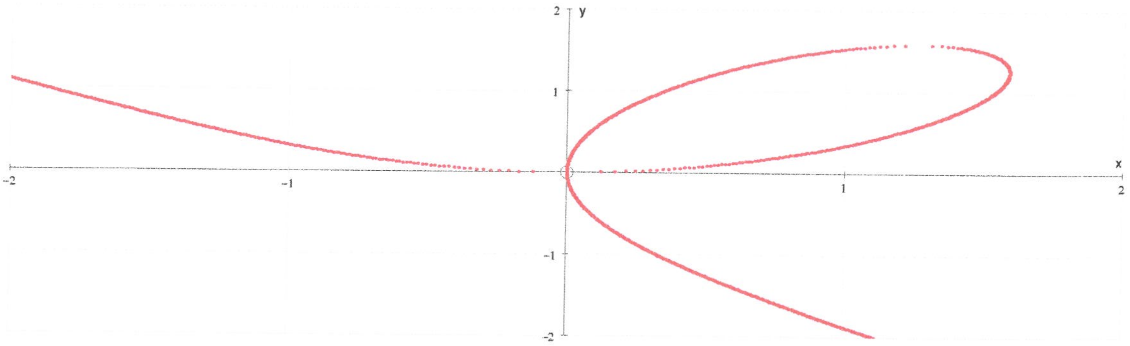
Now $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{-\cancel{4}^{2}t}{\cancel{(1+t^2)^2}} \times \frac{\cancel{(1+t^2)^2}}{\cancel{2}(1-t^2)}$

$= \frac{-2t}{1-t^2} \times \frac{-1}{-1}$

$= \frac{2t}{t^2-1}$

Implicit Differentiation



The graph above plots the equation $x^3 + y^3 = 3xy$, known as *the folium of Descartes*. This is an example of an implicit function – one that cannot be written with y as the subject. Such functions would pose problems with our current level of calculus knowledge if we were required to find an expression for its gradient. However with a little extension of our chain rule work, implicit functions can be differentiated.

Let's start with a more straightforward implicit function

$$x^2 + y^2 + 4x - 6y = 12$$

in order to derive an expression for $\frac{dy}{dx}$ we need to differentiate each term with respect to (w.r.t.) *x*, ie:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(4x) - \frac{d}{dx}(6y) = \frac{d}{dx}(12)$$

Differentiating functions of *x* w.r.t. *x* is straightforward. The problem comes when trying to differentiate a function of *y* w.r.t. *x*. This is where the chain rule is used.

Suppose $z = y^2$, it follows that $\frac{dz}{dy} = 2y$

but if we require $\frac{dz}{dx}$, then using the chain rule:

$$\frac{dz}{dx} = \frac{dz}{d☁} \times \frac{d☁}{dx}$$, where in this instance ☁ $= y$

$$\therefore \frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx} \Rightarrow \frac{dz}{dx} = 2y\frac{dy}{dx}$$

Notice what we have just done – in order to differentiate y^2 w.r.t. *x*, we have differentiated y^2 w.r.t. *y* and then multiplied by $\frac{dy}{dx}$, i.e.

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

We can generalise this as follows:

To differentiate a function of y with respect to x, we differentiate with respect to y and then multiply by $\frac{dy}{dx}$

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}$$

So returning to our circle:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(4x) - \frac{d}{dx}(6y) = \frac{d}{dx}(12)$$

$$2x + 2y\frac{dy}{dx} + 4 - 6\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} + 2 - 3\frac{dy}{dx} = 0$$

$$x + 2 = 3\frac{dy}{dx} - y\frac{dy}{dx}$$

$$\frac{dy}{dx}(3 - y) = x + 2$$

$$\frac{dy}{dx} = \frac{x+2}{3-y}$$

Often, it is also necessary to employ the product rule in order to complete the differentiation.

<u>Eg3</u> Find an expression for the gradient function for the folium of Descartes and hence find the coordinates of the stationary points.

<u>**Exercise 3.6B**</u> (1st Column)

1 Find an expression in terms of x and y for $\frac{dy}{dx}$, given that:

a $x^2 + y^3 = 2$ **b** $x^2 + 5y^2 = 14$ **c** $x^2 + 6x - 8y + 5y^2 = 13$

d $y^3 + 3x^2y - 4x = 0$ **e** $3y^2 - 2y + 2xy = x^3$ **f** $x = \frac{2y}{x^2 - y}$

g $(x - y)^4 = x + y + 5$ **h** $e^x y = xe^y$ **i** $\sqrt{(xy)} + x + y^2 = 0$

2 Find the equation of the tangent to the curve with implicit equation $x^2 + 3xy^2 - y^3 = 9$ at the point $(2, 1)$.

3 Find the equation of the normal to the curve with implicit equation $(x + y)^3 = x^2 + y$ at the point $(1, 0)$.

4 Find the coordinates of the points of zero gradient on the curve with implicit equation $x^2 + 4y^2 - 6x - 16y + 21 = 0$.

Answers

1 a $-\frac{2x}{3y^2}$ **b** $-\frac{x}{5y}$ **c** $\frac{x+3}{4-5y}$

d $\frac{4-6xy}{3x^2+3y^2}$ **e** $\frac{3x^2-2y}{6y+2x-2}$ **f** $\frac{3x^2-y}{2+x}$

g $\frac{4(x-y)^3-1}{1+4(x-y)^3}$ **h** $\frac{e^y-ye^x}{e^x-xe^y}$ **i** $-\frac{(2\sqrt{xy}+y)}{(4y\sqrt{xy}+x)}$

2 $9y + 7x = 23$

3 $y = 2x - 2$

4 $(3, 1)$ and $(3, 3)$

Eg3

$$x^3 + y^3 = 3xy \qquad — ①$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(3xy) \qquad — ②$$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

$$\frac{d}{dx}(3xy) = 3x \cdot 1\frac{dy}{dx} + y \cdot 3 \quad \Leftarrow \text{Product Rule}$$

$$= 3x\frac{dy}{dx} + 3y$$

$\therefore$ ② becomes $\cancel{3}x^2 + \cancel{3}y^2\frac{dy}{dx} = \cancel{3}x\frac{dy}{dx} + \cancel{3}y$

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Stat pts @ $\frac{dy}{dx} = 0 \quad \therefore \quad y - x^2 = 0$

$$y = x^2 \qquad — ③$$

in ① $\quad x^3 + (x^2)^3 = 3x(x^2)$

$$x^3 + x^6 = 3x^3$$

$$x^6 - 2x^3 = 0$$

$$x^3(x^3 - 2) = 0$$

$\therefore$ either $x^3 = 0 \Rightarrow x = 0$ in ③ $y = 0 \Rightarrow (0,0)$

or $x^3 = 2 \Rightarrow x = \sqrt[3]{2} = 2^{1/3}$ in ③ $y = (2^{1/3})^2 = 2^{2/3} = \sqrt[3]{4}$

$$\Rightarrow (\sqrt[3]{2}, \sqrt[3]{4})$$

Ex3.6B

(1) (a) $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}(2)$

$$2x + 3y^2\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{3y^2}$$

(b) $\frac{d}{dx}(x^2) + \frac{d}{dx}(5y^2) = 14$

$$2x + 10y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{10y} = \frac{-x}{5y}$$

(c) $\frac{d}{dx}(x^2) + \frac{d}{dx}(6x) - \frac{d}{dx}(8y) + \frac{d}{dx}(5y^2) = \frac{d}{dx}(13)$

$$2x + 6 - 8\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$$

$\div 2$

$$x + 3 - 4\frac{dy}{dx} + 5y\frac{dy}{dx} = 0$$

$$4\frac{dy}{dx} - 5y\frac{dy}{dx} = x+3$$

$$\frac{dy}{dx}(4-5y) = x+3$$

$$\frac{dy}{dx} = \frac{x+3}{4-5y}$$

Differentiating exponential functions (NOT e^x)

Eg4 Differentiate $y = a^x$, where a is constant

Learn to spot & apply this method.

Take ln's $\ln y = \ln a^x$

$\ln y = x \ln a$ const

diff implicitly $\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \ln a]$

$\frac{1}{y}\frac{dy}{dx} = \ln a \Rightarrow \frac{dy}{dx} = y \ln a$

but $y = a^x$

$\therefore \frac{dy}{dx} = a^x \ln a$

i.e. $\boxed{\text{if } y = a^x, \text{ then } \frac{dy}{dx} = a^x \ln a}$

Exercise 3.6C

1 Find $\frac{dy}{dx}$ for each of the following:

a $y = 3^x$ **b** $y = (\frac{1}{2})^x$ **c** $y = xa^x$ **d** $y = \frac{2^x}{x}$

2 Find the equation of the tangent to the curve $y = 2^x + 2^{-x}$ at the point $(2, 4\frac{1}{4})$.

3 A particular radioactive isotope has an activity R millicuries at time t days given by the equation $R = 200(0.9)^t$. Find the value of $\frac{dR}{dt}$, when $t = 8$.

4 The population of Cambridge was 37 000 in 1900 and was about 109 000 in 2000. Find an equation of the form $P = P_0k^t$ to model this data, where t is measured as years since 1900. Evaluate $\frac{dP}{dt}$ in the year 2000. What does this value represent?

Answers

1 a $3^x \ln 3$ **b** $(\frac{1}{2})^x \ln(\frac{1}{2})$

c $a^x(1 + x \ln a)$ **d** $\frac{2^x(x \ln 2 - 1)}{x^2}$

2 $4y = 15 \ln 2(x - 2) + 17$

3 -9.07 millicuries/day

4 $P = 37\,000k^t$ where $k = \sqrt[100]{\frac{109}{37}}$

1178 people per year

Rate of increase of population during the year 2000.

Ex3C

(1) (c) $y = x a^x$

Product Rule let $u = x$ $\quad V = a^x$

$\frac{du}{dx} = 1$ $\quad \frac{dV}{dx} = a^x \ln x$

$$\frac{dy}{dx} = a^x . 1 + x . a^x \ln x$$

$$= a^x [1 + x \ln x]$$

(d) $y = \frac{2^x}{x}$

Quotient Rule $U = 2^x$ $\quad V = x$

$\frac{du}{dx} = 2^x \ln 2$ $\quad \frac{dV}{dx} = 1$

$$\frac{dy}{dx} = \frac{x 2^x \ln 2 - 2^x . 1}{x^2}$$

$$= \frac{2^x [x \ln 2 - 1]}{x^2}$$

(2) $y = 2^x + 2^{-x}$

$\cancel{\frac{dy}{dx}}$ $\frac{d}{dx}(y) = \frac{d}{dx}(2^x) + \frac{d}{dx}(2^{-x})$

✓ ✓ chain rule

$\cancel{\frac{d}{dx}(2^{-x})} =$ $z = 2^{-x}$

$\ln z = \ln 2^{-x}$

$\ln z = -x \ln 2$

$\frac{1}{z}\frac{dz}{dx} = -\ln 2$

$\frac{dz}{dx} = -z \ln 2$

$\frac{dz}{dx} = -2^{-x} \ln 2$

$\therefore \frac{dy}{dx} = 2^x \ln 2 - 2^{-x} \ln 2$

@ $(2, \frac{17}{4})$ $\frac{dy}{dx} = 2^2 \ln 2 - 2^{-2} \ln 2$

$= 4 \ln 2 - \frac{1}{4} \ln 2$ $3\frac{3}{4}$

$\cancel{= 11 \ln 2}$

$= \frac{15}{4} \ln 2$

eq^n of tangent $y - \frac{17}{4} = \frac{15}{4} \ln 2 (x - 2)$

$\times 4$ $4y - 17 = 15x \ln 2 - 30 \ln 2$

$4y - 15x \ln 2 = 17 - 30 \ln 2$

Using the Chain Rule to link Rates of Change

A further application of the chain rule enables us to combine various rates of change, to produce a differently related rate of change. This is best illustrated using examples!

Eg5 The radius of a circular inkblot is increasing at a rate of 0.3 cm s^{-1}. Find, in $cm^2 s^{-1}$ to 2 sig figs, the rate at which the area of the inkblot is increasing at the instant when the radius of the blot is 0.8 cm.

Eg6 Given that $P = x(x^2 + 4)^{1/2}$, find $\frac{dP}{dt}$ when $x = 2$ and $\frac{dx}{dt} = 3$

Eg7 The edges of a cube are of length x cm. Given that the volume of the cube is being increased at a rate of p $cm^3 s^{-1}$, where p is a constant, calculate, in terms of p, in $cm^2 s^{-1}$, the rate at which the surface area of the cube is increasing when $x = 5$.

Exercise 3.6D

1 Given that $V = \frac{1}{3}\pi r^3$ and that $\frac{dV}{dt} = 8$, find $\frac{dr}{dt}$ when $r = 3$.

2 Given that $A = \frac{1}{4}\pi r^2$ and that $\frac{dr}{dt} = 6$, find $\frac{dA}{dt}$ when $r = 2$.

3 Given that $y = xe^x$ and that $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = 2$.

4 Given that $r = 1 - 3\cos\theta$ and that $\frac{d\theta}{dt} = 3$, find $\frac{dr}{dt}$ when $\theta = \frac{\pi}{6}$.

Answers

1 $\frac{8}{9\pi}$ 2 6π 3 $15e^2$ 4 $\frac{9}{2}$

Eg5 Need $\frac{dA}{dt}$

other variable is radius

So $\frac{dA}{dt} = \frac{dA}{d\square} \times \frac{d\square}{dt}$ where $\square = r$

ie $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

Now $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

And given $\frac{dr}{dt} = 0.3$

$\therefore \frac{dA}{dt} = 2\pi r \times 0.3 = 0.6\pi r$

when $r = 0.8$ $\frac{dA}{dt} = 0.6\pi \times 0.8 = 0.48\pi$

$= 1.5\,cm^2 s^{-1}$ to 2 s.f.

Eg6 $P = x(x^2+4)^{\frac{1}{2}}$

$\frac{dP}{dt} = \frac{dP}{d\square} \times \frac{d\square}{dt}$ where $\square = x$

$\frac{dP}{dt} = \frac{dP}{dx} \times \frac{dx}{dt}$

Product Rule for $\frac{dP}{dx} = x \times \frac{1}{\cancel{2}}(x^2+4)^{-\frac{1}{2}} . \cancel{2}x + (x^2+4)^{\frac{1}{2}} . 1$

$= \frac{\cancel{2}x^2}{\sqrt{x^2+4}} + \sqrt{x^2+4}$

Given $\frac{dx}{dt} = 3$

Eg6 Contd

$$\frac{dP}{dt} = \frac{3x^2}{\sqrt{x^2+4}} + 3\sqrt{x^2+4}$$

when $x = 2$, $\frac{dP}{dt} = \frac{3(4)}{\sqrt{8}} + 3\sqrt{8}$

$$= \frac{12}{\sqrt{8}} + 3\sqrt{8}$$

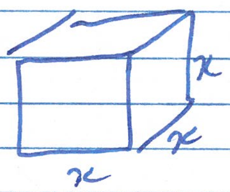
$$= \frac{12+24}{\sqrt{8}}$$

$$= \frac{36}{\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}}$$

$$= \frac{36\sqrt{8}}{8}$$

$$= \frac{9\sqrt{8}}{2} = \frac{9}{\cancel{2}} \times \cancel{2}\sqrt{2} = 9\sqrt{2}$$

Eg7



Need rate of surface area increasing $\frac{dS}{dt} = \frac{dS}{d[illegible]} \times \frac{d[illegible]}{dt}$

where [illegible] $= V$

So will need $\frac{dS}{dt} = \frac{dS}{dV} \times \frac{dV}{dt}$

Now $V = x^3$ — (1)

$S = 6x^2$ — (2)

From (1) [illegible] $x = V^{\frac{1}{3}}$

Eg7 contd n ②

$$S = 6\left(V^{\frac{1}{3}}\right)^2$$

$$S = 6V^{2/3}$$

$$\frac{dS}{dV} = 6 \times \frac{2}{3} V^{-\frac{1}{3}}$$

$$\frac{dS}{dV} = 4V^{-\frac{1}{3}}$$

So $\frac{dS}{dt} = 4V^{-\frac{1}{3}} \times p$

Now when $x=5$, $V = 5^3 = 125$

$$\frac{dS}{dt} = \frac{4}{\sqrt[3]{125}}p = \frac{4p}{5} \text{ cm}^2\text{s}^{-1}$$

5.

OR $\frac{dS}{dt} = \frac{dS}{d\omega} \times \frac{d\omega}{dt}$ where $\omega = V$

$$\frac{dS}{dt} = \frac{dS}{dV} \times \frac{dV}{dt} \leftarrow \text{given} = p.$$

Need $\frac{dS}{dV}$,

$\frac{dS}{dV} = \frac{dS}{d\omega} \times \frac{d\omega}{dV}$ where $\omega = x$ $\quad S = 6x^2 \quad V = x^3$

$\frac{dS}{dx} = 12x \quad \frac{dV}{dx} = 3x^2$

$$\frac{dS}{dV} = \frac{dS}{dx} \times \frac{dx}{dV} = 12x \times \frac{1}{3x^2} = \frac{4}{x}$$

$$\therefore \frac{dS}{dt} = \frac{4}{x}p$$

when $x=5$, $\frac{dS}{dt} = \frac{4}{5}p$

Ex 3.6D

(1) $\frac{dr}{dt} = \frac{dr}{d\omega} \times \frac{d\omega}{dt}$ $\qquad \omega = V$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

given $\frac{dV}{dt} = 8$ and $V = \frac{1}{3}\pi r^3$

$\frac{dV}{dr} = \pi r^2$

$\therefore \frac{dr}{dt} = \frac{1}{\pi r^2} \times 8 = \frac{8}{\pi r^2}$

when $r = 3$ $\quad \frac{dr}{dt} = \frac{8}{9\pi}$

(2) $\frac{dA}{dt} = \frac{dA}{d\omega} \times \frac{d\omega}{dt}$, $\qquad \omega = r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

given $\frac{dr}{dt} = 6$ and $A = \frac{1}{4}\pi r^2$

$\frac{dA}{dr} = \frac{\pi r}{2}$

$\therefore \frac{dA}{dt} = \frac{\pi r}{2} \times 6 = 3\pi r$

when $r = 2$, $\frac{dA}{dt} = 6\pi$

(3) $\frac{dy}{dt} = \frac{dy}{d\square} \times \frac{d\square}{dt}$, where $\square = x$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

given $\frac{dx}{dt} = 5$ and $y = xe^x$

$\frac{dy}{dx} = xe^x + e^x = e^x(x+1)$

$\therefore \frac{dy}{dt} = 5e^x(x+1)$

when $x = 2$, $\frac{dy}{dt} = 15e^2$

(4) $\frac{dr}{dt} = \frac{dr}{d\square} \times \frac{d\square}{dt}$ where $\square = \theta$

$\frac{dr}{dt} = \frac{dr}{d\theta} \times \frac{d\theta}{dt}$

given $\frac{d\theta}{dt} = 3$ and $r = 1 + 3\cos\theta$

$\frac{dr}{d\theta} = -3\sin\theta$

$\therefore \frac{dr}{dt} = -9\sin\theta$

when $\theta = \frac{\pi}{6}$, $\frac{dr}{d\theta} = -\frac{9}{2}$