Integrating with Partial Fractions

Before we consider such integrals, we need to remind ourselves how to integrate functions such as

eg1 $\int \frac{x}{\sqrt{x^2+5}}\,dx$

tinyurl.com/IntPFsEg1

and

eg2 $\int \frac{x+2}{x^2+4x-3}\,dx$

tinyurl.com/IntPFsEg2

from our Core 3 work.

Now an integral such as

$$\int \frac{x-2}{x^2-x-6}\,dx$$

does not appear very different from that in eg 2. However, as the differential of the bottom, will not cancel with the top half of the fraction, we cannot use a substitution method.

As we can factorise the denominator however, we can use partial fractions to make the integration possible.

tinyurl.com/IntPFsEg3

eg3 $\int \frac{x-2}{x^2-x-6}\,dx$

eg4 *Evaluate* $\int_3^4 \frac{2x+1}{(x^2+1)(x-2)}\,dx$, *hence show that this equals* $\ln\left|\frac{2\sqrt{170}}{17}\right|$

tinyurl.com/IntPFsEg4

<u>Exercise 10B</u> Qs>4

Eg1 $\int \frac{x}{\sqrt{x^2+5}}\,dx$

let $u = x^2+5$

$\frac{du}{dx} = 2x$

$\frac{du}{2x} = dx$

$I = \int \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{du}{2\cancel{x}}$

$I = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$I = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{1} u^{\frac{1}{2}}$

$I = (x^2+5)^{\frac{1}{2}} + C$

Eg2 $\displaystyle\int \frac{x+2}{x^2+4x-3}\,dx$

let $u = x^2+4x-3$

$$\frac{du}{dx} = 2x+4$$

$$dx = \frac{du}{2(x+2)}$$

$$I = \int \frac{\cancel{x+2}}{u} \cdot \frac{du}{2(\cancel{x+2})}$$

$$I = \frac{1}{2}\int \frac{1}{u}\,du$$

$$I = \frac{1}{2}\ln|u| + c$$

$$I = \frac{1}{2}\ln|x^2+4x-3| + c$$

eg3 $\displaystyle\int \frac{x-2}{x^2-x-6}\,dx$

$$I = \int \frac{x-2}{(x-3)(x+2)}\,dx$$

Using partial fractions

$$\frac{x-2}{(x-3)(x+2)} \equiv \frac{A}{x-3} + \frac{B}{x+2}$$

$$\equiv \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$$

equivalent numerators

$$\therefore \quad x-2 \equiv A(x+2) + B(x-3)$$

let $x = 3$

$$3-2 = A(3+2)$$

$$1 = 5A$$

$$A = \frac{1}{5}$$

let $x = -2$

$$-2-2 = -5B$$

$$\frac{-4}{-5} = B$$

$$B = \frac{4}{5}$$

$$\therefore \quad I = \int \frac{1}{5(x-3)} + \frac{4}{5(x+2)}\,dx$$

$$= \frac{1}{5}\int \frac{1}{x-3}\,dx + \frac{4}{5}\int \frac{1}{x+2}\,dx$$

$$I = \frac{1}{5}\ln(x-3) + \frac{4}{5}\ln(x+2) + c$$

$$= \frac{1}{5}\ln(x-3).(x+2)^4 + c$$

eg4 $I = \int_3^4 \frac{2x+1}{(x^2+1)(x-2)} \, dx$

Using Partial Fractions

$$\frac{2x+1}{(x^2+1)(x-2)} \equiv \frac{Ax+B}{x^2+1} + \frac{C}{x-2}$$

$$\equiv \frac{(Ax+B)(x-2) + C(x^2+1)}{(x^2+1)(x-2)}$$

equivalent numerators

$$2x+1 \equiv (Ax+B)(x-2) + C(x^2+1)$$

let $x=2$

$$5 = 0 + 5C$$

$$C = 1$$

let $x=0$

$$1 = -2B + 1$$

$$0 = -2B$$

$$B = 0$$

let $x=1$

$$3 = A(-1) + 1(2)$$

$$3 = -A + 2$$

$$A = -1$$

$$\therefore \quad I = \int_3^4 \frac{1}{x-2} - \frac{x}{x^2+1} \, dx$$

$$I = \int_3^4 \frac{1}{x-2} \, dx - \int_3^4 \frac{x}{x^2+1} \, dx$$

let $u = x^2+1$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

$$\int \frac{\cancel{x}}{u} \, \frac{du}{2\cancel{x}}$$

$$\frac{1}{2} \int \frac{1}{u} \, du$$

$$\frac{1}{2} \ln u$$

$$I = \left[\ln(x-2) - \frac{1}{2} \ln(x^2+1) \right]_3^4$$

$$I = \left(\ln 2 - \frac{1}{2} \ln 17 \right) - \left(\cancel{\ln 1} - \frac{1}{2} \ln 10 \right)$$

$$= \ln 2 - \ln \sqrt{17} + \ln \sqrt{10}$$

$$= \ln \left(\frac{2\sqrt{10}}{\sqrt{17}} \right)$$

$$= \ln \left(\frac{2\sqrt{10}}{\sqrt{17}} \times \frac{\sqrt{17}}{\sqrt{17}} \right)$$

$$= \ln \left(\frac{2\sqrt{170}}{17} \right)$$