The General Binomial Expansion

At AS we were introduced to the binomial series $(1 + x)^n$, where n is a positive integer.

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + x^n$$

which can also be written as

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

Eg1 Expand $(1 - 2x)^3$ $= 1 + 3(-2x) + \frac{(3)(2)(-2x)^2}{2!} + \frac{(3)(2)(1)(-2x)^3}{3!}$

$x = (-2x)$

$n = 3$

$= 1 - 6x + 12x^2 - 8x^3$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

We also learned how to use this expansion to expand expressions of the form $(a + b)^n$

For small integer values of n, the coefficients can helpfully be determined using Pascal's Triangle.

Eg2 Expand $(2x + 3)^4$ $= 1(2x)^4(3)^0 + 4(2x)^3(3)^1 + 6(2x)^2(3)^2 + 4(2x)^1(3)^3 + (1)(2x)^0(3)^4$

$= 16x^4 + 4\times8\times3x^3 + 6\times4\times9x^2 + 4\times2\times27x^1 + 1\times81x^0$

$= 16x^4 + 96x^3 + 216x^2 + 216x^1 + 81$

In the above examples, where n is a positive integer, the series will terminate after (n + 1) terms. At A2 we need to apply the expansion when n is **any real number**. In such cases the expansion still holds, but with two important differences:

- the series produced is infinite
- is only valid for $|x| < 1$, *the proof of which is beyond the scope of A Level, but we shall see this is true when considering the behaviour graphically in the examples below.*

Eg3 (a) Expand $(1 - 3x)^{-\frac{2}{3}}$ up to and including the term x^3.

(b) State the set of values of x for which the expansion is valid

(c) Using Geogebra (www.geogebra.org) or similar graph plotting software, consider the validity of the restrictions on x.

(d) Calculate the relative error between the actual function and its approximation when x = 0.1

Unlike when dealing with n as a positive integer, when $n \in \mathbb{R}$, it is not straightforward to expand expressions $(a + b)^n$ without some rearrangement first.

Eg4 Expand $\frac{1}{\sqrt{4-x}}$ in ascending powers of x up to and including the term in x^3, stating the set of values for which the series is valid.

In instances where two or more binomial expansions are used together, if they impose different restrictions on the values of x, then the strictest set of restrictions need to be applied.

Eg5 Find the first three terms in the binomial expansion of $\frac{3}{(1-x)(1+2x)}$

State the set of values for which the expansion is valid.

Exercise 7A Pg 164 Q1 if you wish, 2 to 7

Eg3 (a) $(1-3x)^{-\frac{2}{3}}$

let $x = (-3x)$, $n = \frac{-2}{3}$

$$(1-3x)^{-\frac{2}{3}} = 1 + \left(\frac{-2}{3}\right)(-3x) + \left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\frac{(-3x)^2}{2!} + \left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{-8}{3}\right)\frac{(-3x)^3}{3!} + \ldots$$

$$= 1 + 2x + \frac{90}{18}x^2 + \frac{80 \times 27}{27 \times 6}x^3$$

$$= 1 + 2x + 5x^2 + \frac{40x^3}{3} + \ldots$$

(b) Valid for $|-3x| < 1$ $\quad -1 < -3x < 1$

$\div -3$

$|x| < \frac{1}{3}$ $\quad \frac{1}{3} > x > \frac{-1}{3}$

(c) for $x = 0.1$ $\quad (1 - 3(0.1))^{-\frac{2}{3}} = 1.2684\ldots$ (A)

$1 + 2(0.1) + 5(0.1)^2 + \frac{40}{3}(0.1)^3 = 1.26\dot{3}$ (B)

Relative Error $= \frac{A-B}{A} \times 100 = 0.4\%$

Eg4 $\frac{1}{\sqrt{4-x}} = (4-x)^{-\frac{1}{2}} = \left[4\left(1-\frac{x}{4}\right)\right]^{-1/2}$

$$= 4^{-1/2}\left(1-\frac{x}{4}\right)^{-1/2}$$

$$= \frac{1}{\sqrt{4}}\left(1-\frac{x}{4}\right)^{-1/2}$$

$$= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

let $x = \left(-\frac{x}{4}\right)$, $n = \left(-\frac{1}{2}\right)$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left[1 + \left(\frac{-1}{2}\right)\left(\frac{-x}{4}\right) + \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-x}{4}\right)^2 \times \frac{1}{2!} + \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-x}{4}\right)^3 \times \frac{1}{3!}\right]$$

$$= \frac{1}{2}\left[1 + \frac{x}{8} + \frac{3}{4\times16\times2}x^2 + \frac{15}{8\times64\times6}x^3\right]$$

$$= \frac{1}{2}\left[1 + \frac{x}{8} + \frac{3}{128}x^2 + \frac{\cancel{1}5}{1024}x^3\right]$$

$$= \frac{1}{2} + \frac{x}{16} + \frac{3}{256}x^2 + \frac{5}{2048}x^3$$

Valid for $\left|\frac{-x}{4}\right| < 1$ $\qquad -1 < \frac{-x}{4} < 1$

$|x| < 4$ $\qquad x - 4$

$$+4 > x > -4$$

Eg 5 $\frac{3}{(1-x)(1+2x)} = 3(1-x)^{-1}(1+2x)^{-1}$

Consider $(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \dots$

$x = (-x)$

$n = (-1)$ $\quad = 1 + x + x^2$

Valid for $|-x| < 1 \Rightarrow |x| < 1$

Consider $(1+2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)(+2x)^2}{2!} + \dots$

$x = (2x)$

$n = (-1)$ $\quad = 1 - 2x + 4x^2$

Valid for $|2x| < 1$

$2|x| < 1$

$|x| < \frac{1}{2}$

$\therefore 3(1-x)^{-1}(1+2x)^{-1} = 3(1+x+x^2)(1-2x+4x^2)$

$= (3+3x+3x^2)(1-2x+4x^2)$

$= 3 - 6x + 12x^2 + 3x - 6x^2 + (\dots) + 3x^2 + (\dots)$

$= 3 - 3x + 9x^2$

Values of x need to meet both restriction ie $-1 < x < +1$

and $-\frac{1}{2} < x < \frac{1}{2}$

ie valid for $|x| < \frac{1}{2}$