Partial Fractions

Until this point, any instruction to simplify an algebraic fractional expression was asking you to give the expression as a single fraction. However, there are a number of instances in advanced maths where it is easier to deal with two or three simple separate fractions that it is to handle one more complicated one.

Consider the two-part expression

$$\frac{2}{x+5}+\frac{3}{x+4}$$

Using our C4 skills, we can write this as a single fraction:

$$\frac{2(x+4)+3(x+5)}{(x+5)(x+4)}$$

$$\frac{2x+8+3x+15}{(x+5)(x+4)} \Rightarrow \frac{5x+23}{(x+5)(x+4)}$$

The reverse process of this, ie starting with the single fraction and ending up with the two-part expression is called expressing the algebraic fraction in ***partial fractions***.

The starting point for this process is dependent on what type of denominators the fraction has:

Linear factors in the denominator: (ax + b)(cx + d)

Eg1 Express $\frac{5x+1}{(x-1)(x+2)}$ as a sum of partial fractions.

In some cases, it is necessary to factorise the denominator before finding the partial fractions.

$\frac{7}{4}$ $1\frac{3}{4}$

Eg2 Express $\frac{4x-2}{x^3-x}$ as a sum of partial fractions.

As this demonstrates, the denominator can contain more than two linear factors.

Care must be taken to ensure that the degree of the denominator exceeds that of the numerator, otherwise the fraction is improper (top heavy). In which case, you would have to divide the denominator into the numerator before trying to split the fraction into its partial fractions.

Consider the following

proper! order 1 → order 2 → $\frac{2x+3}{3x^2-2x+1}$ order 2 $\frac{3x^2+2x+6}{5x^2-7}$ order 2 improper order 3 $\frac{2x^3+3x^2-7}{5x-3}$ order 1 Improper

Exercise 7D Pg 176 Primes

$5x^2-7\,\overline{)\,3x^2+2x+6}$ $5x-3\,\overline{)\,2x^3+3x^2-7}$

Eg1 $$\frac{5x+1}{(x-1)(x+2)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

$$\frac{5x+1}{(x-1)(x+2)} \equiv \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

Considering numerators

$$5x+1 \equiv A(x+2)+B(x-1)$$

METHOD 1 let $x=1$ $\quad 5(1)+1 = A(1+2) + B(0)$

$$6 = 3A$$

$$A = 2$$

let $x=-2$ $\quad 5(-2)+1 = (0)A + B(-2-1)$

$$-9 = -3B$$

$$B = 3$$

METHOD 2 $\quad 5x+1 \equiv Ax + 2A + Bx - B$

Compare coefficients $\quad x^1: \quad A+B = 5 \quad -(1)$

$x^0: \quad 2A - B = 1 \quad -(2)$

(1) + (2) $\quad 3A = 6$

$A = 2$

Sub in (1) $\quad 2+B = 5$

$B = 3$

$$\therefore \frac{5x+1}{(x-1)(x+2)} \equiv \frac{2}{x-1} + \frac{3}{x+2}$$

Eg2 $\frac{4x-2}{x^3-x} \equiv \frac{4x-2}{x(x^2-1)} \equiv \frac{4x-2}{x(x+1)(x-1)}$

$$\frac{4x-2}{x(x+1)(x-1)} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$4x-2 \equiv A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

Method 1 $x=1$ $\quad 2 = 2C$

$C = 1.$

$x=-1$ $\quad -6 = B(-1)(-2)$

$-6 = 2B$

$B = -3$

$$4x-2 = A(x+1)(x-1) - 3x(x-1) + x(x+1)$$

let $x=2$

$6 = A(3)(1) - 3(2)(1) + 2(3)$

$6 = 3A - 6 + 6$

$A = 2$

Method 2 $\quad 4x-2 \equiv A(x^2-1) + B(x^2-x) + C(x^2+x)$

Compare Coefficients

$x^2: \quad A + B + C = 0 \quad — (1)$

$x^1: \quad -B + C = 4 \quad — (2)$

$x^0: \quad -A = -2 \quad — (3)$

$A = 2$

from (2) $\quad B = C - 4 \quad — (4)$

Sub in (1) $\quad 2 + C - 4 + C = 0$

$2c - 2 = 0$

$2c = 2$

$c = 1$

h (4) $B = 1 - 4 = -3$

$$\therefore \frac{4x-2}{x^3-x} = \frac{2}{x} - \frac{3}{x+1} + \frac{1}{x-1}$$

Denominators containing a quadratic factor

If any of the factors in the denominator is not linear, then the partial fractions cannot take the form demonstrated in the previous examples.

In cases where the denominator contains a factor which is a non-reducible quadratic then the partial fraction takes a different form.

Eg3 Express $\frac{5x^2+4x+4}{(x^2+4)(x+2)}$ in partial fractions

Repeated Factors in the Denominator

This is where one of the factors has been squared. In such instances the partial fraction takes another form.

Eg4 Express $\frac{2(x^2-2x-1)}{(x+1)(x-1)^2}$ as partial fractions.

Exercise 7E Pg178, Q1 (i) to (iv)

Using Partial Fractions with the Binomial Expansion

We can use binomial expansions to produce simplified approximations to some complex algebraic fractions if they can be written as partial fractions.

Eg5 Given the function $f(x) = \frac{x}{(3-2x)(2-x)}$

✓a. Express f(x) in partial fractions.
✓b. Expand f(x) in ascending powers of x to produce a quadratic approximation.
✓c. State the range of values of x for which this expansion is valid.
d. Calculate the relative error in this approximation when x = 0.1

Exercise 7D Pg 180, Q1 if you wish (note you have already found the partial fractions in Ex7E), 2, 4 & 4

Eg3 $\dfrac{5x^2+4x+4}{(x^2+4)(x+2)} \equiv \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x+2}$

$5x^2+4x+4 \equiv (Ax+B)(x+2) + C(x^2+4)$

$5x^2+4x+4 \equiv Ax^2+2Ax+Bx+2B+Cx^2+4C$

Compare Coefficients

$x^2: \quad A+C = 5 \quad -(1)$

$x^1: \quad 2A+B = 4 \quad -(2)$

$x^0: \quad 2B+4C = 4 \quad -(3)$

from (1) $\quad A = 5-C \quad -(4)$

from (3) $\quad B+2C = 2$

$B = 2-2C \quad -(5)$

Sub in (2) $\quad 2(5-C)+2-2C = 4$

$10-2C+2-2C = 4$

$12-4C = 4$

$4C = 8$

$C = 2$

in (4) $\quad A = 3$

in (5) $\quad B = -2$

$\therefore$ equivalent $\quad \dfrac{3x-2}{x^2+4} + \dfrac{2}{x+2}$

Eg4 $\dfrac{2(x^2-2x-1)}{(x+1)(x-1)^2} \equiv \dfrac{A}{x+1} + \dfrac{B}{(x-1)} + \dfrac{C}{(x-1)^2}$

$2x^2-4x-2 \equiv A(x-1)^2 + B(x+1)(x-1) + C(x+1)$

$\equiv A(x^2-2x+1) + B(x^2-1) + C(x+1)$

Compare Coefficients

x^2: $A + B = 2$ — (1)

x^1: $-2A + C = -4$ — (2)

x^0: $A - B + C = -2$ — (3)

From (1) $A = 2 - B$ — (4)

In (2) $-2(2-B) + C = -4$

$-4 + 2B + C = -4$

$C = -2B$ — (5)

In (3) $2 - B - B - 2B = -2$

$2 - 4B = -2$

$-4B = -4$

$B = 1$

In (4) $A = 1$

In (5) $C = -2$

$\therefore$ equivalence $\dfrac{1}{x+1} + \dfrac{1}{x-1} \dfrac{-2}{(x-1)^2}$

Eg5 (a) $f(x) = \dfrac{x}{(3-2x)(2-x)} \equiv \dfrac{A}{3-2x} + \dfrac{B}{2-x}$

$$x \equiv A(2-x) + B(3-2x)$$

Compare Coeficient

$x^1: \quad -A - 2B = 1 \quad —(1)$

$x^0: \quad 2A + 3B = 0 \quad —(2)$

From (1) $\quad A = -2B - 1 \quad —(3)$

In (2) $\quad 2(-2B-1) + 3B = 0$

$$-4B - 2 + 3B = 0$$
$$-B = 2$$
$$B = -2$$

In (3) $\quad A = -2(-2) - 1 = 3$

$\therefore f(x) = \dfrac{3}{3-2x} - \dfrac{2}{2-x}$

(b) $f(x) = 3(3-2x)^{-1} - 2(2-x)^{-1}$

$$= 3\left[3\left(1 - \frac{2x}{3}\right)\right]^{-1} - 2\left[2\left(1 - \frac{x}{2}\right)\right]^{-1}$$

$$= 3\left[3^{-1}\left(1 - \frac{2x}{3}\right)^{-1}\right] - 2\left[2^{-1}\left(1 - \frac{x}{2}\right)^{-1}\right]$$

$$= \cancel{3}\left[\frac{1}{\cancel{3}}\left(1 - \frac{2x}{3}\right)^{-1}\right] - \cancel{2}\left[\frac{1}{\cancel{2}}\left(1 - \frac{x}{2}\right)^{-1}\right]$$

$$= \left(1 - \frac{2x}{3}\right)^{-1} - \left(1 - \frac{x}{2}\right)^{-1}$$

expanding

$$\left(1-\frac{2}{3}x\right)^{-1} = 1 + (-1)\left(-\frac{2x}{3}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{2x}{3}\right)^2 + \dots$$

$$= 1 + \frac{2x}{3} + \frac{4}{9}x^2 + \dots$$

Valid for $\left|-\frac{2x}{3}\right| < 1$

$$\frac{2}{3}|x| < 1$$

$$|x| < \frac{3}{2} \quad \text{or} \quad -\frac{3}{2} < x < \frac{3}{2}$$

expanding

$$\left(1-\frac{x}{2}\right)^{-1} = 1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 + \dots$$

$$= 1 + \frac{x}{2} + \frac{x^2}{4} + \dots$$

Valid for $\left|-\frac{x}{2}\right| < 1$

$$\frac{1}{2}|x| < 1$$

$$|x| < 2 \quad \text{or} \quad -2 < x < 2$$

Hence $f(x) \approx \left(1 + \frac{2x}{3} + \frac{4}{9}x^2\right) - \left(1 + \frac{x}{2} + \frac{x^2}{4}\right)$

$$\approx \frac{x}{6} + \frac{7}{36}x^2$$

(c) Valid for strictest of two ranges, $|x| < \frac{3}{2}$

(d) when $x = 0.1$ $f(x) = \frac{0.1}{(3-2(0.1))(2-0.1)} = \frac{5}{266}$ $(\Rightarrow A)$

$$\approx \frac{0.1}{6} + \frac{7}{36}(0.1)^2 = \frac{67}{3600} \quad (\Rightarrow B)$$

$$\% \text{ error} = \frac{B-A}{A} \times 100 = 0.99\%$$