Complex Exponents

When multiplying complex numbers in modulus-argument form you add the arguments, and when multiplying powers of the same base you add the exponents. This suggests that there may be a link between the modulus-argument expression

$$z = r(\cos\theta + i\sin\theta)$$

which we first met in FP1 and the exponential function.

This was first noticed in 1714 by an English mathematician called Roger Cotes and made widely know through a book published by Euler in 1748.

In a later chapter of FP2, Maclaurin's expansion will be used to produce the following series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^r}{r!} + \ldots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

$i = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$i^6 = -1$
$i^7 = -i$
$\vdots$

Now if x is the complex number iθ, the first series becomes

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots + \frac{(i\theta)^r}{r!} + \ldots$$

which leads to an important result:

$$e^{i\theta} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \ldots}_{\Downarrow \ \cos\theta} + i\underbrace{\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right]}_{\Downarrow \ \sin\theta}$$

$$\therefore e^{i\theta} = \cos\theta + i\sin\theta$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

If we compare the above result with the modulus-argument form at the top of the page, we can write a complex number in a third form:

$$\boxed{z = re^{i\theta}}$$

where $r = |z|$ and $\theta = \arg z$. This is called the exponential form of a complex number z.

As an aside, if you consider the particular case where $\theta = \pi$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$e^{i\pi} = -1 + 0$$

$$e^{i\pi} + 1 = 0$$

This remarkable statement linking the five fundamental numbers 0, 1, i, e, π, the three fundamental operations of addition, multiplication and exponentiation, and the fundamental relation of equality has been described as a 'mathematical poem'!

Eg1 Write the following in the form (i) $r(\cos\theta + i\sin\theta)$ and (ii) $re^{i\theta}$

(a) $\sqrt{3} - 3i$ (b) 8 (c) $-2 - 2i$ (d) $-i$

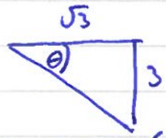
Eg2 Write the following in the form a + ib

(a) $5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (b) $\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$

Exercise 3A Page 23

Eg 1 (a) $Z = \sqrt{3} - 3i$

$r = |z| = \sqrt{(\sqrt{3})^2 + (-3)^2} = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$



$\cancel{\theta = \frac{3}{\sqrt{3}}}$ $\theta = \tan^{-1}\left(\frac{3}{\sqrt{3}}\right) = \frac{-\pi}{3}$

(i) $Z = 2\sqrt{3}\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$

(ii) $Z = 2\sqrt{3}\, e^{-i\frac{\pi}{3}}$

(b) $Z = 8$

$r = |z| = 8 \qquad \theta = 0$

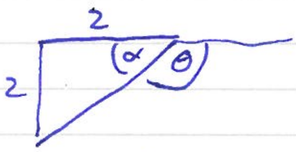
(i) $Z = 8(\cos 0 + i\sin 0)$

(ii) $Z = 8e^{i0}$

(c) $Z = -2 - 2i$

$r = |z| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

θ



$\theta = -\pi + \frac{\pi}{4} = \frac{-3\pi}{4}$

(1) $Z = 2\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

(11) $Z = 2\sqrt{2}\, e^{-i\frac{3\pi}{4}}$

(d) $Z = -i$

$r = 1 \qquad \theta = \frac{-\pi}{2}$

(1) $Z = 1\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$

(11) $Z = 1e^{\frac{-i\pi}{2}}$

Eg2 (a) $Z = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$Z = 5\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \frac{5}{2} + \frac{i5\sqrt{3}}{2}$

(b) $Z = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$

$Z = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

(c) $Z = \dfrac{\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2}{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)} = \dfrac{\left(\cos\frac{\pi}{4}\times 2 + i\sin\frac{\pi}{4}\times 2\right)}{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}$

$= \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$

$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= \frac{1}{2} + \frac{i\sqrt{3}}{2}$

Ex3A

① (a) $7 = 7(\cos 0 + i\sin 0)$

(b) $-5i = 5\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

(c) $\sqrt{3}+i \qquad r = \sqrt{4} = 2 \qquad \theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$= 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

(d) $2+2i \qquad r = 2\sqrt{2} \qquad \theta = \frac{\pi}{4}$

$= 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

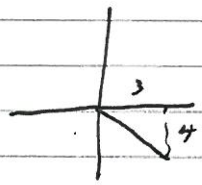
(e) $1-i \qquad r = \sqrt{2} \qquad \theta = -\frac{\pi}{4}$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

(f) $-8 = 8(\cos\pi + i\sin\pi)$

(g) $3-4i \qquad r = 5 \qquad \theta = \arctan\left(\frac{4}{3}\right) = \bar{0}.93^c$

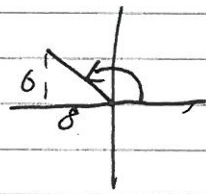
$= 5\left(\cos(\bar{0}.93) + i\sin(\bar{0}.93^c)\right)$



(h) $-8+6i \qquad r = 10 \qquad \theta = \pi - \arctan\left(\frac{6}{8}\right)$

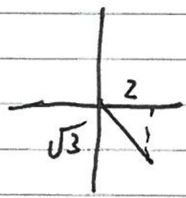
$= 2.50^c$

$= 10\left(\cos(2.50) + i\sin(2.50)\right)$



(i) $2-\sqrt{3}i \qquad r = \sqrt{7} \qquad \theta = \arctan\left(\frac{\sqrt{3}}{2}\right) = 0.71$

$= \sqrt{7}\left(\cos(-0.71) + i(-0.71)\right)$



(2) (a) $5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 5(0+i) = 5i$

(b) $\frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{1}{2}\left(\frac{\sqrt3}{2} + i\frac{1}{2}\right) = \frac{\sqrt3}{4} + \frac{1}{4}i$

(c) $6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 6\left(-\frac{\sqrt3}{2} + i\frac{1}{2}\right) = -3\sqrt3 + 3i$

(d) $3\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right) = 3\left(-\frac{1}{2} + -\frac{\sqrt3 i}{2}\right) = -\frac{3}{2} - \frac{3\sqrt3}{2}i$

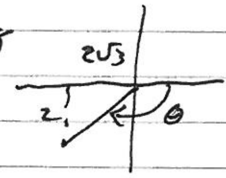
(e) $2\sqrt2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\sqrt2\left(\frac{1}{\sqrt2} - \frac{1}{\sqrt2}i\right) = 2 - 2i$

(f) $-4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -4\left(-\frac{\sqrt3}{2} - \frac{1}{2}i\right) = 2\sqrt3 + 2i$

(3). (a) $-3 = 3e^{\pi i}$

(b) $6i = 6e^{\frac{\pi}{2}i}$

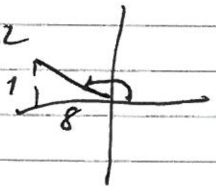
(c) $-2\sqrt3 - 2i$ $\quad r = 4$ $\quad \arctan\left(\frac{2}{2\sqrt3}\right) = \frac{\pi}{6}$



$\theta = -\frac{5\pi}{6}$

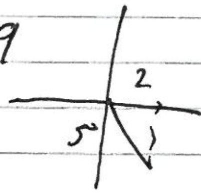
$4e^{-\frac{5\pi}{6}i}$

(d) $-8 + i$ $\quad r = \sqrt{65}$ $\quad \theta = \pi - \arctan\left(\frac{1}{8}\right) = 3.02$



$\therefore \sqrt{65}e^{3.02i}$

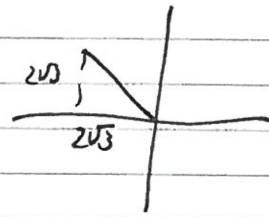
(e) $2 - 5i$ $\quad r = \sqrt{29}$ $\quad \theta = -\arctan\left(\frac{5}{2}\right) = -1.19$



$\sqrt{29}e^{-1.19i}$

(f) $-2\sqrt3 + 2\sqrt3 i$ $\quad r = 2\sqrt6$ $\quad \theta = +\frac{3\pi}{4}$

$2\sqrt6 e^{\frac{3\pi}{4}i}$



③ (g) $\sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{8}e^{\frac{\pi}{4}i} = 2\sqrt{2}e^{\frac{\pi}{4}i}$

(h) $8\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 8\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 8e^{-\frac{\pi}{6}i}$

(i) $2\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right) = 2\left(\cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)\right) = 2e^{-\frac{\pi}{5}i}$

④ (a) $e^{\frac{\pi}{3}i} = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

(b) $4e^{\pi i} = 4\left(\cos\pi + i\sin\pi\right) = -4$

(c) $3\sqrt{2}e^{\frac{\pi}{4}i} = 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 3\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 3+3i$

(d) $8e^{\frac{\pi}{6}i} = 8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 8\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 4\sqrt{3} + 4i$

(e) $3e^{-\frac{\pi}{2}i} = 3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = 3(0 - i) = -3i$

(f) $e^{\frac{5\pi}{6}i} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(g) $e^{-\pi i} = \cos -\pi + i\sin -\pi = -1.$

(h) $3\sqrt{2}e^{-\frac{3\pi}{4}i} = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = 3\sqrt{2}\left(-\frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}}i\right)$

$= -3 - 3i$

(i) $8e^{-\frac{4\pi}{3}i} = 8\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right) = 8\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -4 + 4\sqrt{3}i$

(5)

(a) $e^{\frac{16\pi}{13}i} = 1\left(\cos\frac{16\pi}{13} + i\sin\frac{16\pi}{13}\right)$ $\qquad \frac{16\pi}{13} - 2\pi = -\frac{10}{13}$

$= \cos\left(-\frac{10}{13}\right) + i\sin\left(-\frac{10}{13}\right)$

(b) $4e^{\frac{17\pi}{5}i}$ $\qquad \frac{17\pi}{5} - 2\pi = \frac{7\pi}{5} - 2\pi = -\frac{3\pi}{5}$

$= 4\left(\cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)\right)$

(c) $5e^{-\frac{9\pi}{8}i} =$ $\qquad -\frac{9\pi}{8} + 2\pi = \frac{7\pi}{8}$

$= 5\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right)$

(6)

$e^{i\theta} = \cos\theta + i\sin\theta \quad — (1)$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \quad — (2)$

(1) − (2) $\quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\therefore \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ As required.

Multiplying and Dividing Complex Numbers in Modulus-Argument Form

Consider the complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \left[\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\right]$$

$$= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$$

$$\cos(\theta_1+\theta_2) \qquad\qquad \sin(\theta_1+\theta_2)$$

$$\therefore\ z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

It can similarly be shown that $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$

In exponential form:

$$z_1 z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

<u>Eg3</u> Given that $z = 2\left[\cos\left(\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{\pi}{4}\right)\right]$ and $w = 3\left[\cos\left(\dfrac{\pi}{3}\right) + i\sin\left(\dfrac{\pi}{3}\right)\right]$ express in the form $x + iy$

(i) wz

(ii) $\dfrac{w}{z}$

(iii) 5iz

<u>Exercise 3B</u> Pg 27

Eg3 (i) $wz = 6\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right]$

$= 6\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right]$

$= 6\left[-0.26 + 0.97i\right]$

$= -1.56 + 5.8i$

(ii) $\frac{w}{z} = \frac{3}{2}\left[\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right]$

$= 1.5\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$= 1.4 + 0.4i$

(iii) $5iz = 5i\left[2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]$

$= 10i\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right]$

$= \frac{10}{\sqrt{2}}i - \frac{10}{\sqrt{2}}$

$= -5\sqrt{2} + 5\sqrt{2}i$

Ex 3B

(1) a) $\cos(2\theta+3\theta) + i\sin(2\theta+3\theta) = \cos 5\theta + i\sin 5\theta$

(b) $\cos\left(\frac{3\pi}{11}+\frac{8\pi}{11}\right) + i\sin\left(\frac{3\pi}{11}+\frac{8\pi}{11}\right) = \cos\pi + i\sin\pi = -1$

(c) $3(2)\left(\cos\left(\frac{\pi}{4}+\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{12}\right)\right) = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 6\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 3 + 3\sqrt{3}i$

(d) $\sqrt{6}\times\sqrt{3}\left(\cos\left(-\frac{\pi}{12}+\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{12}+\frac{\pi}{3}\right)\right) = 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 3\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= 3 + 3i$

(e) $4\times\frac{1}{2}\left(\cos\left(-\frac{5\pi}{9}-\frac{5\pi}{18}\right) + i\sin\left(-\frac{5\pi}{9}-\frac{5\pi}{18}\right)\right) = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$

$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= -\sqrt{3} - i$

(f) $6\times5\times\frac{1}{3}\left(\cos\left(\frac{\pi}{10}+\frac{\pi}{3}+\frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{10}+\frac{\pi}{3}+\frac{2\pi}{5}\right)\right) = 10\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 10\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$= -5\sqrt{3} + 5i$

(g) $(\cos 4\theta + i\sin 4\theta)(\cos\theta - i\sin\theta)$

$= (\cos 4\theta + i\sin 4\theta)(\cos(-\theta) + i\sin(-\theta)) = \cos 3\theta + i\sin 3\theta$

(h) $3\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \times \sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$= 3\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 3\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 3 - 3i$

② (a) $\cos 3\theta + i\sin 3\theta$

(b) $\frac{\sqrt{2}}{\frac{1}{2}}\left(\cos\left(\frac{\pi}{2}-\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\right) = 2\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$

$= 2+2i$

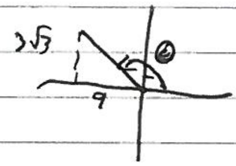
(c) $\frac{3}{4}\left(\cos\left(\frac{\pi}{3}-\frac{5\pi}{6}\right)+i\sin\left(\frac{\pi}{3}-\frac{5\pi}{6}\right)\right) = \frac{3}{4}\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

$= -\frac{3}{4}i$

(d) $\frac{\cos 2\theta - i\sin 2\theta}{\cos 3\theta + i\sin 3\theta} = \frac{\cos(-2\theta)+i\sin(-2\theta)}{\cos 3\theta + i\sin 3\theta} = \cos(-5\theta)+i\sin(-5\theta)$

③ $z = -9+3\sqrt{3}i$ $\quad |w| = \sqrt{3}$ $\quad \arg w = \frac{7\pi}{12}$

(a) $r = 6\sqrt{3}$



$\theta = \pi - \arctan\left(\frac{3\sqrt{3}}{9}\right)$

$= \pi - \arctan\left(\frac{1}{\sqrt{3}}\right)$

$= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$z = 6\sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right)$

(b) $w = \sqrt{3}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$

(c) $zw = 6\sqrt{3}\times\sqrt{3}\left(\cos\left(\frac{17\pi}{12}\right)+i\sin\left(\frac{17\pi}{12}\right)\right)$ $\quad \frac{17\pi}{12}-2\pi = -\frac{7\pi}{12}$

$= 18\left(\cos\left(\frac{-7\pi}{12}\right)+i\sin\left(\frac{-7\pi}{12}\right)\right)$

(d) $\frac{z}{w} = \frac{6\sqrt{3}}{\sqrt{3}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = 6\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$.

De Moivre's Theorem

By extending the property of multiplying two complex numbers in modulus-argument form, ie for two numbers z_1 and z_2;

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

we can consider the effect of raising a single complex number to a power:

if $z = r(\cos\theta + i\sin\theta)$

it follows that

$z^2 = r \times r\left(\cos(\theta+\theta) + i\sin(\theta+\theta)\right) = r^2(\cos 2\theta + i\sin 2\theta)$

$z^3 = r \times r \times r\left(\cos(\theta+\theta+\theta) + i\sin(\theta+\theta+\theta)\right) = r^3(\cos 3\theta + i\sin 3\theta)$

which implies that

$z^n = r^n(\cos n\theta + i\sin n\theta)$

This relationship was first considered by Abraham De Moivre in the 18th Century, the proof of which needs to be learned.

If $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

then $z^n = [re^{i\theta}]^n = r^n e^{in\theta}$

Now $e^{in\theta} = \cos n\theta + i\sin n\theta$

$\therefore\ z^n = r^n(\cos n\theta + i\sin n\theta)$

Eg4 Find the value of $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{12}$

Eg5 If $z = 1 - i\sqrt{3}$, find (i) z^4 and (ii) $\frac{1}{z^3}$

Exercise 3C Pg 31

De Moivre's Theorem

By extending the property of multiplying two complex numbers in modulus-argument form, ie for two numbers z_1 and z_2;

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

we can consider the effect of raising a single complex number to a power:

if $z = r(\cos\theta + i\sin\theta)$

it follows that

$z^2 = r \times r\left(\cos(\theta+\theta) + i\sin(\theta+\theta)\right) = r^2(\cos 2\theta + i\sin 2\theta)$

$z^3 = r \times r \times r\left(\cos(\theta+\theta+\theta) + i\sin(\theta+\theta+\theta)\right) = r^3(\cos 3\theta + i\sin 3\theta)$

which implies that

$z^n = r^n(\cos n\theta + i\sin n\theta)$

This relationship was first considered by Abraham De Moivre in the 18th Century, the proof of which needs to be learned.

If $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

then $z^n = [re^{i\theta}]^n = r^n e^{in\theta}$

Now $e^{in\theta} = \cos n\theta + i\sin n\theta$

$\therefore\ z^n = r^n(\cos n\theta + i\sin n\theta)$

Eg4 Find the value of $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{12}$

Eg5 If $z = 1 - i\sqrt{3}$, find (i) z^4 and (ii) $\frac{1}{z^3}$

Exercise 3C Pg 31

Eg4 $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{12} = \cos\left(\frac{\pi}{4}\times 12\right) + i\sin\left(\frac{\pi}{4}\times 12\right)$

$= \cos 3\pi + i\sin 3\pi$

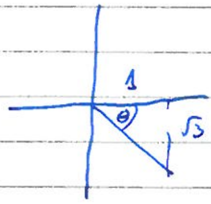
$= -1 + i0$

$= -1.$

Eg5 $z = 1 - i\sqrt{3}$

Convert into Mod-arg Form:

$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$



$\arg = \tan^{-1}\left(\sqrt{3}\right) = -\frac{\pi}{3}$

$\therefore \quad z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

Now (i) $z^4 = 2^4\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right)$

$= 16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -8 + i8\sqrt{3}$

(ii) $\frac{1}{z^3} = z^{-3} = 2^{-3}\left(\cos\left(-3\times -\frac{\pi}{3}\right) + i\sin\left(-3\times -\frac{\pi}{3}\right)\right)$

$= \frac{1}{8}\left(\cos\pi + i\sin\pi\right)$

$= \frac{1}{8}\left(-1 + i0\right)$

$= -\frac{1}{8}.$

Ex3C

(1) (a) $(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

(b) $(\cos 3\theta + i\sin 3\theta)^4 = \cos 12\theta + i\sin 12\theta$

(c) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(d) $\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

(e) $\left(\cos 2\pi + i\sin\pi\right) = 1$

(f) $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15} = \left(\cos\left(-\frac{\pi}{10}\right) + i\sin\left(-\frac{\pi}{10}\right)\right)^{15} = \cos(-1.5\pi) + i\sin(-1.5\pi) = i$

(g) $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 2\theta + i\sin 2\theta)^2} = \cos\theta + i\sin\theta$

(h) $\dfrac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3} = \cos(14\theta - 12\theta) + i\sin(14\theta - 12\theta) = \cos 2\theta + i\sin 2\theta$

(i) $\dfrac{1}{(\cos 2\theta + i\sin 2\theta)^3} = (\cos 2\theta + i\sin 2\theta)^{-3} = \cos(-6\theta) + i\sin(-6\theta)$
$= \cos 6\theta - i\sin 6\theta$

(j) $\dfrac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

(k) $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta - i\sin 3\theta)^2} = \dfrac{\cos 5\theta + i\sin 5\theta}{(\cos(-3\theta) + i\sin(-3\theta))^2} = \dfrac{\cos 5\theta + i\sin 5\theta}{\cos(-6\theta) + i\sin(-6\theta)}$

$= \cos 11\theta + i\sin 11\theta$ * Answer diff in book

(l) $\dfrac{\cos\theta - i\sin\theta}{(\cos 2\theta - i\sin 2\theta)^3} = \dfrac{\cos(-\theta) + i\sin(-\theta)}{\cos(-6\theta) + i\sin(-6\theta)} = \cos 5\theta + i\sin 5\theta$

(2) $$\frac{\left(\cos\frac{7\pi}{13}+i\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13}-i\sin\frac{4\pi}{13}\right)^6} = \frac{\cos\frac{28\pi}{13}+i\sin\frac{28\pi}{13}}{\cos\left(-\frac{24\pi}{13}\right)+i\sin\left(-\frac{24\pi}{13}\right)}$$

$$= \cos 4\pi + i\sin 4\pi$$

$$= 1.$$

(3)(a) $z = (1+i)^5$

$r = \sqrt{2}$

$\theta = \frac{\pi}{4}$

$$\therefore\ z = \sqrt{2}^5\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^5 = (\sqrt{2})^5\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$$

$$= (\sqrt{2})^5\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$= -4-4i$$

(b) $z = (-2+2i)^8$

$r = 2\sqrt{2}$

$\theta = \frac{3\pi}{4}$

$$\therefore\ z = 2\sqrt{2}^8\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)^8 = 4096(\cos 6\pi + i\sin 6\pi)$$

$$= 4096$$

(c) $z = (1-i)^6$

$r = \sqrt{2}$

$\theta = -\frac{\pi}{4}$ $\quad\therefore\ z = \sqrt{2}^6\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)^6$

$$= 8\left(\cos(-1.5\pi)+i\sin(-1.5\pi)\right)$$

$$= 8i$$

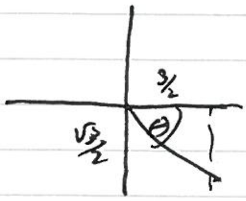
③(d) $Z = (1-\sqrt{3}i)^6$

$r = \cancel{4}2$

$\theta = -\arctan(\sqrt{3}) = -\frac{\pi}{3}$

$\therefore Z = 2^6\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)^6 = \cancel{4096}\ 64$

(e) $Z = \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9$



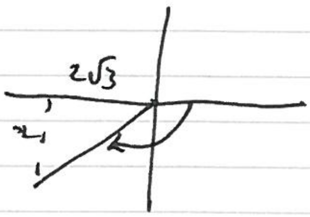
$r = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$

$\theta = -\arctan\left(\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$

$\therefore Z = \sqrt{3}^9\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)^9 = 81\sqrt{3}\left(\cos(-1.5\pi) + i\sin(-1.5\pi)\right)$

$= 81\sqrt{3}i$

(f) $Z = (-2\sqrt{3} - 2i)^5$



$r = \sqrt{12 + 4} = 4$

$\theta = -\pi + \arctan\left(\frac{2}{2\sqrt{3}}\right) = -\frac{5\pi}{6}$

$\therefore Z = 4^5\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)^5$

$= 1024\left(\cos\left(-\frac{25\pi}{6}\right) + i\sin\left(-\frac{25\pi}{6}\right)\right)$

$= 1024\left(\frac{\sqrt{3}}{2} \cancel{+} - \frac{1}{2}i\right)$

$= 512\sqrt{3} - 512i$

(4) $z = (3+\sqrt{3}i)^5$

$r = \sqrt{9+3} = 2\sqrt{3}$

$\theta = \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

$\therefore \ z = (2\sqrt{3})^5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

$= 288\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 288\sqrt{3}\left(-\frac{\sqrt{3}}{2} + \frac{1i}{2}\right)$

$= -432 + 144\sqrt{3}\,i$

De Moivre's theorem can also be used to obtain certain types of trigonometric identity.

Eg6 Express cos 3θ in terms of cos θ and tan 3θ in terms of tan θ.

Exercise 3D Pg 36 Q's 1, 2, 3

Expressions for powers of sin θ and cos θ in terms of sines and cosines of multiples of θ can be derived using the following results:

If $z = \cos\theta + i\sin\theta$, then $\frac{1}{z} = (\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

hence

$z + \frac{1}{z} = 2\cos\theta$ and $z - \frac{1}{z} = 2i\sin\theta$

By De Moivre's theorem,

$z^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{z^n} = \cos n\theta - i\sin n\theta$

so

$$\boxed{z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta}$$

Eg7 Express $\sin^4\theta$ in terms of cosines of multiples of θ. Hence find $\int \sin^4\theta \, d\theta$.

Exercise 3D Pg 36 Q's 4, 5, 6, 7

Eg6 $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

LHS

$$\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$

$$(\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) \quad — 1$$

Compare Real parts

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$

$$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$

$$= 4\cos^3\theta - 3\cos\theta$$

Now for $\tan 3\theta$: imag 1 $\div$ Real 1

$$\tan 3\theta = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

$$\times \frac{\frac{1}{\cos^3\theta}}{\frac{1}{\cos^3\theta}}$$

$$\tan 3\theta = \frac{\frac{3\cos^2\theta\sin\theta}{\cos^3\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta\sin^2\theta}{\cos^3\theta}}$$

$$\therefore \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Eg 7 Need $Sn^4\theta$ $\therefore$ use $(2iSin\theta)^4 = \left(z - \frac{1}{z}\right)^4$

$$16Sn^4\theta = 1.z^4\left(-\frac{1}{z}\right)^0 + 4z^3\left(-\frac{1}{z}\right)^1 + 6z^2\left(-\frac{1}{z}\right)^2 + 4z^1\left(-\frac{1}{z}\right)^3 + 1.z^0\left(-\frac{1}{z}\right)^4$$

$$= z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 - \frac{1}{z^2}\right) + 6$$

Now $16Sn^4\theta = 2Cos4\theta - 4.2Cos2\theta + 6$

$$16Sn^4\theta = 2Cos4\theta - 8Cos2\theta + 6$$

$$\therefore \quad Sn^4\theta = \frac{1}{8}Cos4\theta - \frac{1}{2}Cos2\theta + \frac{3}{8}$$

Hence $\int Sn^4\theta \, d\theta$

$$= \int \frac{1}{8}Cos4\theta - \frac{1}{2}Cos2\theta + \frac{3}{8} \, d\theta$$

$$= \frac{1}{32}Sn4\theta - \frac{1}{4}Sn2\theta + \frac{3\theta}{8} + c$$

Ex3D

① $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

1
1 1
1 2 1
1 3 3 1

LHS

$$\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

equating imaginary parts

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$\sin 3\theta = 3\sin\theta(1-\sin^2\theta) - \sin^3\theta$$

$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$

$$= 3\sin\theta - 4\sin^3\theta$$ As required.

② $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

LHS

$$\cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$

$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

equating imaginary parts

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= 5\sin\theta(1-\sin^2\theta)^2 - 10\sin^3\theta(1-\sin^2\theta) + \sin^5\theta$$

$$= 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 11\sin^5\theta$$

$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$ As required.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

(3) $(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$

$\cos 7\theta$ requires ~~even~~ real parts = even powers of sine.

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta) - 7\cos\theta(1-3\cos^2\theta+3\cos^4\theta - \cos^6\theta)$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

As required.

(4). $\left(z + \frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta$

LHS $z^4 + 4\cancel{z^3}^{\,2}\left(\frac{1}{\cancel{z}}\right) + 6\cancel{z^2}\left(\frac{1}{\cancel{z}}\right)^2 + 4\cancel{z}\left(\frac{1}{\cancel{z}^{\,2}}\right)^3 + \left(\frac{1}{z}\right)^4$

$\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$2\cos 4\theta + 4(2\cos 2\theta) + 6$

$\therefore 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$16\cos^4\theta = 2[\cos 4\theta + 4\cos 2\theta + 3]$

$\cos^4\theta = \frac{1}{8}[\cos 4\theta + 4\cos 2\theta + 3]$ As required.

(5) $\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5 = 32i^5\sin^5\theta = 32i\sin^5\theta$

LHS

$z^5 + 5z^4\left(-\frac{1}{z}\right)^1 + 10z^3\left(-\frac{1}{z}\right)^2 + 10z^2\left(-\frac{1}{z}\right)^3 + 5z^1\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$

$= z^5 - 5z^3 + 10z - 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^3}\right) - \left(\frac{1}{z^5}\right)$

$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$

$= 2\cancel{i}\sin5\theta - 5(2\cancel{i}\sin3\theta) + 10(2\cancel{i}\sin\theta) = 32\cancel{i}\sin^5\theta$

$= 2\sin5\theta - 10\sin3\theta + 20\sin\theta = 32\sin^5\theta$

$\sin^5\theta = \frac{\cancel{2}\,1}{\cancel{32}\,16}\left[\sin5\theta - 5\sin3\theta + 10\sin\theta\right]$ As required

$$\begin{array}{c}1\\1\ 1\\1\ 2\ 1\\1\ 3\ 3\ 1\\1\ 4\ 6\ 4\ 1\\1\ 5\ 10\ 10\ 5\ 1\\1\ 6\ 15\ 20\ 15\ 6\ 1\end{array}$$

6 (a) $\left(z+\frac{1}{z}\right)^6 = \left(2\cos\theta\right)^6 = 64\cos^6\theta$

LHS

$$z^6 + 6z^5\left(\frac{1}{z}\right)^1 + 15z^4\left(\frac{1}{z}\right)^2 + 20z^3\left(\frac{1}{z}\right)^3 + 15z^2\left(\frac{1}{z}\right)^4 + 6z\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$$

$$z^6 + 6z^4 + 15z^2 + 20 + 15\left(\frac{1}{z^2}\right) + 6\left(\frac{1}{z^4}\right) + \left(\frac{1}{z^6}\right)$$

$$\left(z^6+\frac{1}{z^6}\right) + 6\left(z^4+\frac{1}{z^4}\right) + 15\left(z^2+\frac{1}{z^2}\right) + 20$$

$$2\cos6\theta + 6(2\cos4\theta) + 15(2\cos2\theta) + 20 = 64\cos^6\theta$$

$\div 2$ $\quad 32\cos^6\theta = \cos6\theta + 6\cos4\theta + 15\cos2\theta + 10$ $\quad$ As required

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \cos^6\theta \, d\theta = \frac{1}{32}\int_0^{\frac{\pi}{6}} \cos6\theta + 6\cos4\theta + 15\cos2\theta + 10 \, d\theta$

$$= \frac{1}{32}\left[\frac{1}{6}\sin6\theta + \frac{6}{4}\sin4\theta + \frac{15}{2}\sin2\theta + 10\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{32}\left[0 + \frac{6}{4}\cdot\frac{\sqrt{3}}{2} + \frac{15\sqrt{3}}{4} + \frac{10\pi}{6}\right]$$

$$= \frac{1}{32}\left[\frac{18\sqrt{3}}{4} + \frac{5\pi}{3}\right]$$

$$= \frac{9\sqrt{3}}{64} + \frac{5\pi}{96}$$

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ 1\ \ 4\ \ 6\ \ 4\ \ 1 \end{array}$$

7 (a) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

<u>LHS</u>

$$\cos^4\theta + 4\cos^3\theta(i\sin\theta)^1 + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

$$(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

Compare imaginary parts

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \quad \text{— 1} \quad \text{As required}$$

(b) Compare real parts

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \quad \text{— 2}$$

$$\cancel{1 \div \cos^4\theta \quad \frac{\sin 4\theta}{\cos^4\theta} = 4\sin\theta - 4\cos\theta\tan^3\theta}$$

$$1 \div \cos^4\theta \quad \frac{\sin 4\theta}{\cos^4\theta} = \frac{4\cos^3\theta\sin\theta}{\cos^4\theta} - 4\frac{\cos\theta\sin^3\theta}{\cos^4\theta}$$

$$\frac{\sin 4\theta}{\cos^4\theta} = 4\tan\theta - 4\tan^3\theta \quad \text{— 3}$$

$$2 \div \cos^4\theta \quad \frac{\cos 4\theta}{\cos^4\theta} = \frac{\cos^4\theta}{\cos^4\theta} - \frac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \frac{\sin^4\theta}{\cos^4\theta}$$

$$\frac{\cos 4\theta}{\cos^4\theta} = 1 - 6\tan^2\theta + \tan^4\theta \quad \text{— 4}$$

Now $3 \div 4$ $\quad \dfrac{\sin 4\theta}{\cos 4\theta} = \tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ As required.

(c) $\tan 4\theta(1 - 6\tan^2\theta + \tan^4\theta) = 4\tan\theta - 4\tan^3\theta$

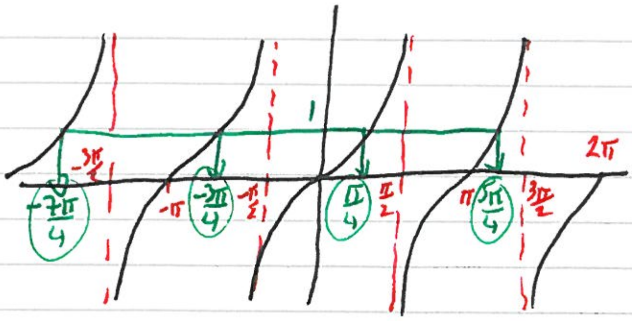
If $\tan 4\theta = 1$ $\Leftarrow$ this choice produces required eq

Then $1 - 6\tan^2\theta + \tan^4\theta = 4\tan\theta - 4\tan^3\theta$

$$\tan^4\theta + 4\tan^3\theta - 6\tan^2\theta - 4\tan\theta + 1 = 0$$

If $\tan\theta = x$ then $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

(7)(c) cont'd now $\tan 4\theta = 1$



$$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{7\pi}{4}$$

$$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, -\frac{3\pi}{16}, -\frac{7\pi}{16}$$

and $x = \tan\theta$

$$= \tan\left(\frac{\pi}{16}\right) = 0.20$$

$$= \tan\left(\frac{5\pi}{16}\right) = 1.50$$

$$= \tan\left(-\frac{3\pi}{16}\right) = -0.67$$

$$= \tan\left(-\frac{7\pi}{16}\right) = -5.03.$$

Complex Roots – The Roots of Unity

As early as 1629 Albert Girard stated that every polynomial equation of degree *n* has exactly *n* roots (including repetitions); this was first proved by the 18 year old Carl Friedrich Gauss 170 years later.

Therefore, even the simple equation $z^n = 1$ has *n* roots. Of course one of these is $z = 1$, and if *n* is even $z = -1$ is another. But where are the rest?

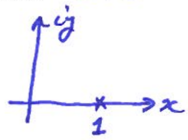
Let us first consider the solution to the equation $z^3 = 1$, ie what are the cube roots of 1?

The cube roots of unity

$\sqrt[3]{1}$ can be written in index form as $(1)^{\frac{1}{3}}$

Now as a complex number, 1 can be written as $1 + 0i$

On an argand diagram, this looks like



and in modulus-argument form, $r(\cos\theta + i\sin\theta)$, where $r = 1$ and $\theta = 0, 2\pi, 4\pi, 6\pi, \ldots 2k\pi$ where $k = 0, 1, 2, 3, \ldots$

So $1 = \cos 2k\pi + i\sin 2k\pi$

and $(1)^{\frac{1}{3}} = (\cos 2k\pi + i\sin 2k\pi)^{1/3}$ $k = 0, 1, 2, 3, \ldots$

By De Moivre's Theorem then $(1)^{\frac{1}{3}} = \left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)$

but according to the work of Girard and Gauss there should only be three roots, so what happens for k = 3, 4, 5, etc?

First root, when $k=0$; $\cos 0 + i\sin 0$ Modulus 1, argument 0

2^{nd} root, when $k=1$: $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ Modulus 1, arg $\frac{2\pi}{3}$

3^{rd} root, when $k=2$: $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ Modulus 1, arg $-\frac{2\pi}{3}$ (remember $-\pi < \theta < \pi$)

~~4th~~, when $k=3$: $\cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}$ Modulus 1, arg 0

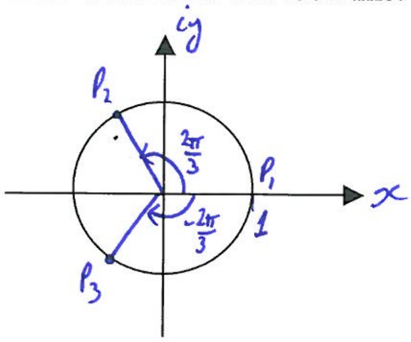
when $k=4$: $\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}$ Modulus 1, arg $\frac{2\pi}{3}$

Same three roots will be repeated again & again

On an argand diagram

Properties of the cube roots of unity

On an Argand diagram, these three roots will look like:



- The points P_1, P_2 and P_3, representing the roots lie on a circle, centred at the origin and with radius 1 and that the angle between each successive point is the same, $\frac{2\pi}{3}$.

- It is customary to use ω to represent the root with the smallest positive argument, P_2 on our Argand diagram.

  So $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

  and again using De Moivre's theorem

  $$\omega^2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = P_3$$

  So the cube roots of unity can be written as 1, ω and ω^2.

- Look again at $\omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

  $$\omega^2 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^* = \omega^*$$

  ie ω is the conjugate of ω^2 and vice-versa

- If we consider the sum of these roots, ie

  $$1 + \omega + \omega^2 = 1 + \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + \left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$$

  $$= 1 + 2\cos\frac{2\pi}{3}$$

  $$= 1 + 2 \times -\frac{1}{2}$$

  $$= 1 - 1$$

  $$= 0$$

  ie sum of cube roots of unity is zero.

The n^{th} roots of unity

We can now extend the theory to consider $\sqrt[n]{1}$:

if $\quad 1 = \cos 2k\pi + i\sin 2k\pi$, where $k = 0, 1, 2, 3, \ldots$

then $\quad (1)^{\frac{1}{n}} = \left(\cos 2k\pi + i\sin 2k\pi\right)^{1/n}$

By DMTh: $\quad (1)^{1/n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ where $k = 0, 1, 2, 3, \ldots, n-1$

(when $k \geq n$, cycle is repeated)

The properties of the n^{th} roots of unity extend from those encountered when considering the cube roots above, namely:

- The first root is always 1.
- If n is even, another root is -1, when $k = \frac{n}{2}$.
- If $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then the n^{th} roots of unity can be written as $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-2}, \omega^{n-1}$
- On an Argand diagram the roots would form the vertices of a regular n-sided polygon, inscribed in the unit circle, with one vertex at the point 1.
- With the exception of 1 (and -1 if n is even) the other roots occur in conjugate pairs, with $\omega = \left(\omega^{n-1}\right)^*$, and $\omega^2 = \left(\omega^{n-2}\right)^*$, etc.
- Consider the sum of the n^{th} roots of unity:

  $1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^{n-2} + \omega^{n-1}$

  This is a geometric series with first term 1 and common ratio ω.

  The sum is $\dfrac{1-\omega^n}{1-\omega}$ $\quad \left\{\text{from C2: } S_n = \dfrac{a(1-r^n)}{1-r}\right\}$

  but as ω is an n^{th} root of 1, then $\sqrt[n]{1} = \omega$, so $\omega^n = 1$

  therefore

  $1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^{n-2} + \omega^{n-1} = \dfrac{1-\omega^n}{1-\omega} = \dfrac{1-1}{1-\omega} = 0$

$\therefore$ Sum of n^{th} roots of unity is zero.

Eg ~~10~~ 8 $\quad$ Find the fifth roots of unity and represent them on an Argand diagram.

Eg 10 $\quad \sqrt[5]{1}$

$\pi = 180$

$\frac{2}{5}\pi = \frac{2}{5} \times 180 = 72°$

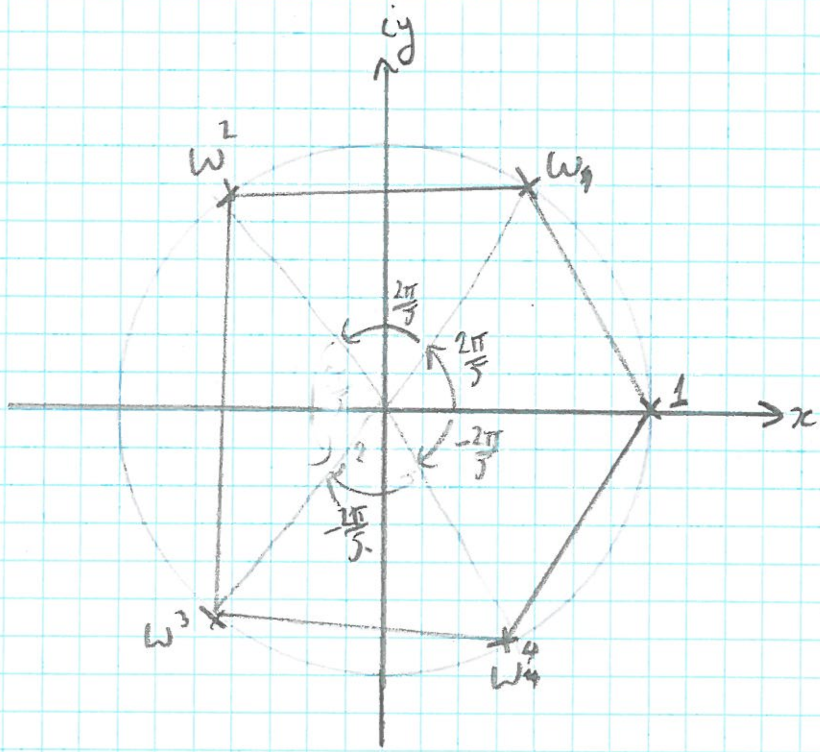
$$(1)^{\frac{1}{5}} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$$

$1,\ \omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} \qquad \arg\ \frac{2\pi}{5}$

$\omega^2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} \qquad \arg\ \frac{4\pi}{5}$

$\omega^3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right) \qquad \arg\ -\frac{4\pi}{5}$

$\omega^4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right) \qquad \arg\ -\frac{2\pi}{5}$



Complex Roots – The General Case

The complex number $z = r(\cos\theta + i\sin\theta)$ can also be written as

$z = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$, where $k = 0, 1, 2, 3, \ldots$

so

$z^{\frac{1}{n}} = \{r(\cos(\theta+2k\pi) + i\sin(\theta+2k\pi))\}^{1/n}$

by De Moivre's theorem

$z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$ $k = 0, 1, 2, \ldots, n-1$

~~Eg11~~ Eg 9 Find in the form $re^{i\theta}$ the five fifth roots of 2 – 2i and plot them on an Argand diagram.

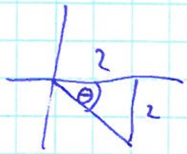
~~Exercise 2B Pg 36 Q's 14 to 17~~ Ex 3E Pg. 40.

~~No additional theory required…~~

~~Exercise 2B Pg 36 Q's 18 to 24.~~

Eg11 if $z = 2-2i$

$$r = \sqrt{2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$$



$$\theta = \frac{-\pi}{4}$$

$$\therefore\ z = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\sqrt{2}\left(\cos\left(\frac{2k\pi-\pi}{4}\right) + i\sin\left(\frac{2k\pi-\pi}{4}\right)\right)$$

$$\cancel{\text{Now } z^{1/5} = (2\sqrt{2})^{1/5}}\quad z = 2\sqrt{2}\left(\cos\left(\frac{8k\pi-\pi}{4}\right) + i\sin\left(\frac{8k\pi-\pi}{4}\right)\right)$$

$$z^{1/5} = (2\sqrt{2})^{1/5}\left(\cos\pi\left(\frac{8k-1}{4}\right) + i\sin\pi\left(\frac{8k-1}{4}\right)\right)^{1/5}$$

$$= (2\sqrt{2})^{1/5}\left(\cos\pi\left(\frac{8k-1}{20}\right) + i\sin\pi\left(\frac{8k-1}{20}\right)\right)$$

$k=0$ 1st Root $(2\sqrt{2})^{1/5} e^{-\pi/20}$ ✓

$k=1$ 2nd Root $(2\sqrt{2})^{1/5} e^{\frac{7\pi}{20}}$ ✓

$k=2$ 3rd Root $(2\sqrt{2})^{1/5} e^{\frac{15\pi}{20}}$ ✓ $= (2\sqrt{2})^{1/5} e^{\frac{3\pi}{4}}$

$k=3$ 4th Root $(2\sqrt{2})^{1/5} e^{\frac{23\pi}{20}}$ ✓ $= (2\sqrt{2})^{1/5} e^{-\frac{17}{20}\pi}$

$k=4$ 5th Root $(2\sqrt{2})^{1/5} e^{\frac{31\pi}{20}} = (2\sqrt{2})^{1/5} e^{-\frac{9\pi}{20}}$

On argand diagram, regular pentagon inscribed in circle radius $(2\sqrt{2})^{1/5} = 1.23$
Vertices @ $-9^\circ, 63^\circ, 135^\circ, -153^\circ, -81^\circ$

Ex3E

(1)(a) $Z^4 - 1 = 0$

$$Z = (1)^{1/4}$$

$$Z = \left[1^{1/4}\left(\cos(0+2k\pi) + i\sin(0+2k\pi)\right)\right]^{1/4}$$

$$Z = \cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)$$

$k=0 \quad Z_1 = 1$

$k=1 \quad Z_2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$k=2 \quad Z_3 = \cos\pi + i\sin\pi = -1$

$k=3 \quad Z_4 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$

(b) $Z^3 = i$

$r = 1 \quad \theta = \frac{\pi}{2}$

$$Z = \left[\cos\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{\pi}{2} + 2k\pi\right)\right]^{1/3}$$

$$Z = \cos\frac{\pi}{6}(1+4k) + i\sin\frac{\pi}{6}(1+4k)$$

$k=0 \quad Z_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$k=1 \quad Z_2 = \cancel{Ai}\ -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$k=2 \quad \cancel{Z_3 = \frac{\sqrt{3}}{2}}\ -i$

1(c) $z^3 = 27$

$r = 27 \quad \theta = 0$

$z = \left[27\left(\cos(0+2k\pi) + i\sin(0+2k\pi)\right)\right]^{1/3}$

$z = 3\left[\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right]$

$\cancel{z = 3}$

$k=0 \quad z_1 = 3$

$k=1 \quad z_2 = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

$k=2 \quad z_3 = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$

(d) $z^4 + 64 = 0$

$z^4 = -64$

$r = 64 \quad \theta = -\pi$

$z = \left[64\left(\cos(-\pi+2k\pi) + i\sin(-\pi+2k\pi)\right)\right]^{1/4}$

$z = 2\sqrt{2}\left[\cos\frac{\pi}{4}(2k-1) + i\sin\frac{\pi}{4}(2k-1)\right]$

$k=0 \quad z_1 = 2\sqrt{2}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = 2-2i$

$k=1 \quad z_2 = 2\sqrt{2}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = 2+2i$

$k=2 \quad z_3 = 2\sqrt{2}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = -2+2i$

$k=3 \quad z_4 = 2\sqrt{2}\left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = -2-2i$

① (a) $z^4 + 4 = 0$

$z^4 = -4$

$r = 4 \quad \theta = -\pi$

$$z = \left[4\left(\cos(-\pi + 2k\pi) + i\sin(-\pi + 2k\pi)\right)\right]^{1/4}$$

$$z = \sqrt{2}\left[\cos\frac{\pi}{4}(2k-1) + i\sin\frac{\pi}{4}(2k-1)\right]$$

$k=0 \quad z_1 = \sqrt{2}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = 1 - i$

$k=1 \quad z_2 = \sqrt{2}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = 1 + i$

$k=2 \quad z_3 = \sqrt{2}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = -1 + i$

$k=3 \quad z_4 = \sqrt{2}\left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = -1 - i$

(b) $z^3 + 8i = 0$

$z^3 = -8i$

$r = 8 \quad \theta = -\frac{\pi}{2}$

$$\therefore z = \left[8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + i\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{1/3}$$

$$z = 2\left[\cos\frac{\pi}{6}(4k-1) + i\sin\frac{\pi}{6}(4k-1)\right]$$

$k=0 \quad z_1 = 2\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = \sqrt{3} - 1i$

$k=1 \quad z_2 = 2\left[0 + i\right] = 2i$

$k=2 \quad z_3 = 2\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = -\sqrt{3} - i$

(2) (a) $z^7 = 1$

$r = 1 \quad \theta = 0$

$z = \left[1(\cos(0+2k\pi) + i\sin(0+2k\pi))\right]^{1/7}$

$z = \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}$

$k=0 \quad z_1 = \cos 0 + i\sin 0$

$k=1 \quad z_2 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$k=2 \quad z_3 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}$

$k=3 \quad z_4 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}$

$k=4 \quad z_5 = \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7} \quad (-2\pi) \quad = \cos\left(-\frac{6\pi}{7}\right) + i\sin\left(-\frac{6}{7}\pi\right)$

$k=5 \quad z_6 = \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7} \quad (-2\pi) = \cos\left(-\frac{4\pi}{7}\right) + i\sin\left(-\frac{4\pi}{7}\right)$

$k=6 \quad z_7 = \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7} \quad (-2\pi) = \cos\left(-\frac{2\pi}{7}\right) + i\sin\left(-\frac{2\pi}{7}\right).$

(b) $z^4 + 16i = 0$

$z^4 = -16i$

$r = 16 \quad \theta = -\frac{\pi}{2}$

$z = \left[16\left(\cos\left(-\frac{\pi}{2}+2k\pi\right) + i\sin\left(-\frac{\pi}{2}+2k\pi\right)\right)\right]^{1/4}$

$z = 2\left[\cos\frac{\pi}{8}(4k-1) + i\sin(4k-1)\right]$

$k=0 \quad z_1 = 2\left[\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right]$

2(b) $k=1 \quad z_2 = 2\left[\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right]$

$k=2 \quad z_3 = 2\left[\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right]$

$k=3 \quad z_4 = 2\left[\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right] \quad (-2\pi) \quad z_4 = 2\left[\cos\left(\frac{-5\pi}{8}\right) + i\sin\left(\frac{-5\pi}{8}\right)\right]$

(c) $z^5 + 32 = 0$

$z^5 = -32$

$r = 32 \quad \theta = -\pi$

$\therefore z = \left[32\left(\cos(-\pi + 2k\pi) + i\sin(-\pi + 2k\pi)\right)\right]^{1/5}$

$z = 2\left[\cos\frac{\pi}{5}(2k-1) + i\sin\frac{\pi}{5}(2k-1)\right]$

$k=0 \quad z_1 = 2\left[\cos\left(\frac{-\pi}{5}\right) + i\sin\left(\frac{-\pi}{5}\right)\right]$

$k=1 \quad z_2 = 2\left[\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right]$

$k=2 \quad z_3 = 2\left[\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right]$

$k=3 \quad z_4 = 2\left[\cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}\right] = 2\left[\cos\pi + i\sin\pi\right]$

$k=4 \quad z_5 = 2\left[\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right] \quad (-2\pi) \quad z_5 = 2\left[\cos\left(\frac{-3\pi}{5}\right) + i\sin\left(\frac{-3\pi}{5}\right)\right]$

② (d) $z^3 = 2+2i$

$r = \sqrt{8} = 2\sqrt{2} \qquad \theta = \frac{\pi}{4}$

$z = \left[2\sqrt{2}\left(\cos\left(\frac{\pi}{4}+2k\pi\right) + i\sin\left(\frac{\pi}{4}+2k\pi\right)\right)\right]^{1/3}$

$z = (2\sqrt{2})^{1/3}\left[\cos\frac{\pi}{12}(8k+1) + i\sin\frac{\pi}{12}(8k+1)\right]$

$(2\sqrt{2})^{1/3} = (2^1 \cdot 2^{1/2})^{1/3}$
$= (2^{3/2})^{1/3}$
$= 2^{1/2}$

$k=0 \quad z_1 = \sqrt{2}\left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]$

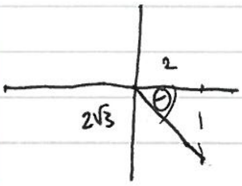
$k=1 \quad z_2 = \sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

$k=2 \quad z_3 = \sqrt{2}\left[\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right] \quad (-2\pi) \quad z_3 = \sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right]$

(e) $z^4 + 2\sqrt{3}i = 2$

$z^4 = 2 - 2\sqrt{3}i$

$r = \sqrt{2^2 + (2\sqrt{3})^2} = 4$



$\theta = \arctan\left(\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

$\therefore z = \left[4\left(\cos\left(-\frac{\pi}{3}+2k\pi\right) + i\sin\left(-\frac{\pi}{3}+2k\pi\right)\right)\right]^{1/4}$

$z = \sqrt{2}\left[\cos\frac{\pi}{12}(6k-1) + i\sin\frac{\pi}{12}(6k-1)\right]^{1/4}$

$k=0 \quad z_1 = \sqrt{2}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]$

$k=1 \quad z_2 = \sqrt{2}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$

② (e) $k=2$ $\quad Z_3 = \sqrt{2}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$

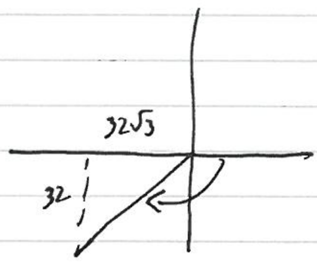
$k=3$ $\quad Z_4 = \sqrt{2}\left[\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right]$ $\quad (-2\pi) \quad Z_4 = \sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right]$

(f) $z^3 + 32\sqrt{3} + 32i = 0$

$z^3 = -32\sqrt{3} - 32i$

$r = \sqrt{(-32\sqrt{3})^2 + (-32)^2} = \sqrt{4096} = 64$

$\theta = -\pi + \arctan\left(\frac{32}{3\sqrt{3}}\right) = -\frac{5\pi}{6}$



$Z = \left[64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$Z = 4\left[\cos\frac{\pi}{18}(12k-5) + i\sin\frac{\pi}{18}(12k-5)\right]$

$k=0$ $\quad Z_1 = 4\left[\cos\left(\frac{-5\pi}{18}\right) + i\sin\left(\frac{-5\pi}{18}\right)\right]$

$k=1$ $\quad Z_2 = 4\left[\cos\left(\frac{7\pi}{18}\right) + i\sin\left(\frac{7\pi}{18}\right)\right]$

$k=2$ $\quad Z_3 = 4\left[\cos\left(\frac{19\pi}{18}\right) + i\sin\left(\frac{19\pi}{18}\right)\right] \approx -2\pi \quad Z_3 = 4\left[\cos\left(-\frac{17\pi}{18}\right) + i\sin\left(-\frac{17\pi}{18}\right)\right]$

3 (a) $z^4 = 3+4i$

$r = 5 \quad \theta = \arctan\left(\frac{4}{3}\right) = 0.93^c$

$z^4 = \left[5\left(\cos(0.93+2k\pi)+i\sin(0.93+2k\pi)\right)\right]^{1/4}$

$z = 5^{1/4}\left[\cos\left(\frac{0.93+2k\pi}{4}\right)+i\sin\left(\frac{0.93+2k\pi}{4}\right)\right] = 5^{1/4}e^{i\left(\frac{0.93+2k\pi}{4}\right)}$

$k=0 \quad z_1 = 5^{1/4}e^{0.23i}$

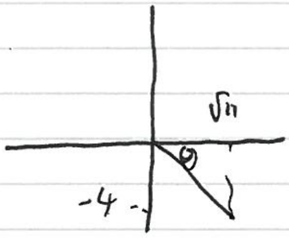
$k=1 \quad z_2 = 5^{1/4}e^{1.80i}$

$k=2 \quad z_3 = 5^{1/4}e^{3.37i} \quad (-2\pi) \quad = 5^{1/4}e^{-2.91i}$

$k=3 \quad z_4 = 5^{1/4}e^{4.94i} \quad (-2\pi) \quad = 5^{1/4}e^{-1.34i}$

(b) $z^3 = \sqrt{11}-4i$

$r = \sqrt{(\sqrt{11})^2+(-4)^2} = \sqrt{27} = 3\sqrt{3}$



$\theta = -\arctan\left(\frac{4}{\sqrt{11}}\right) = -0.89^c$

$z = \left[3\sqrt{3}e^{(-0.89+2k\pi)i}\right]^{1/3}$

$z = (3\sqrt{3})^{1/3}e^{\left(\frac{2k\pi-0.89}{3}\right)i} = \left(3^{3/2}\right)^{1/3}e^{\left(\frac{2k\pi-0.89}{3}\right)} = \sqrt{3}e^{\left(\frac{2k\pi-0.89}{3}\right)}$

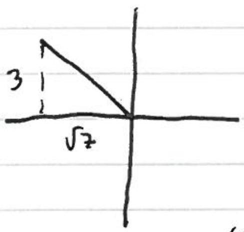
$k=0 \quad z_1 = \sqrt{3}e^{-0.29i}$

$k=1 \quad z_2 = \sqrt{3}e^{1.80i}$

$k=2 \quad z_3 = \sqrt{3}e^{3.90i} \quad (-2\pi) \quad = \sqrt{3}e^{-2.39i}$

③(c) $z^4 = -\sqrt{7} + 3i$

$r = \sqrt{(\sqrt{7})^2 + 3^2} = 4$



$\theta = \pi - \arctan\frac{3}{\sqrt{7}} = 2.29^c$

$z = \left[4e^{(2.29+2k\pi)i}\right]^{1/4} = \sqrt{2}e^{\left(\frac{2.29+2k\pi}{4}\right)i}$

k=0 $z_1 = \sqrt{2}e^{0.57i}$

k=1 $z_2 = \sqrt{2}e^{2.14i}$

k=2 $z_3 = \sqrt{2}e^{3.71i}$ -2π $= \sqrt{2}e^{-2.57i}$

k=3 $z_4 = \sqrt{2}e^{5.29i}$ -2π $= \sqrt{2}e^{1.0i}$

④(a) $(z+1)^3 = -1$

$r = 1$ $\theta = \pi$

$z+1 = \left(\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)\right)^{1/3}$

$z = \left[\cos\left(\frac{\pi+2k\pi}{3}\right) - 1\right] + i\sin\left(\frac{\pi+2k\pi}{3}\right)$

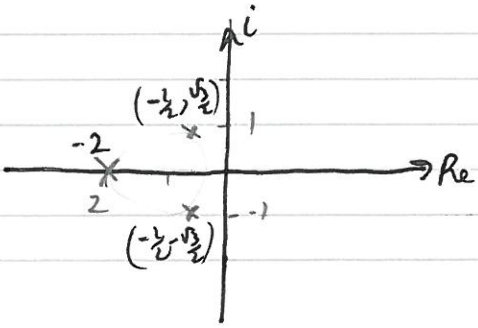
k=0 $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

k=1 $z_2 = -2$

k=2 $z_3 = \cos\left(\frac{5\pi}{3} - 2\pi\right) - 1 + i\sin\left(\frac{5\pi}{3} - 2\pi\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

diff in book

(b)



(c) $z^3 = -1$ whose would have lie on circle centre (0,0), and radius 1.

So $(z+1)^3 = -1$ transforms $\begin{pmatrix}-1\\0\end{pmatrix}$

∴ Circle centre (-1,0) radius 1.

(5) (a) $z^5 - 1 = 0$

$z^5 = 1 \qquad r = 1 \qquad \theta = \cancel{\tfrac{\pi}{2}}\, 0$

$$z = \left[1\left(\cos\left(\cancel{\tfrac{\pi}{2}} + 2k\pi\right) + i\sin\left(\cancel{\tfrac{\pi}{2}} + 2k\pi\right)\right)\right]^{1/5}$$

$$z = \cancel{\cos \tfrac{\pi}{10}(4k+1) + i\sin\tfrac{\pi}{10}(4k+1)} \quad \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)$$

$k=0 \quad z_1 = 1$

$k=1 \quad z_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$k=2 \quad z_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$k=3 \quad z_4 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} \quad -2\pi. \quad = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$

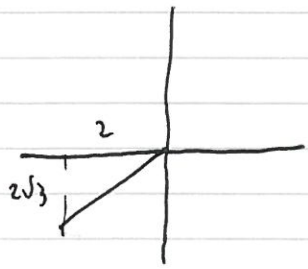
$k=4 \quad z_5 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} \quad -2\pi \quad = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$

$$1 + \left(\cos\frac{2\pi}{5} + \cancel{i\sin\frac{2\pi}{5}}\right) + \left(\cos\frac{4\pi}{5} + \cancel{i\sin\frac{4\pi}{5}}\right) + \left(\cos\frac{4\pi}{5} - \cancel{i\sin\frac{4\pi}{5}}\right) + \left(\cos\frac{2\pi}{5} - \cancel{i\sin\frac{2\pi}{5}}\right) = 0$$

$$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$$

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$$ As required

6 (a) $r = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4$



$\theta = -\pi + \arctan\left(\frac{2\sqrt{3}}{2}\right) = -\frac{2\pi}{3}$

(b) $z^4 + 2 + 2\sqrt{3}i = 0$

$z^4 = -2 - 2\sqrt{3}i$

$z^4 = 4\left(\cos\left(-\frac{2\pi}{3} + 2k\pi\right) + i\sin\left(-\frac{2\pi}{3} + 2k\pi\right)\right)$

$z = 4^{\frac{1}{4}}\left[\cos\frac{2\pi}{3}(3k-1) + i\sin\frac{2\pi}{3}(3k-1)\right]^{\frac{1}{4}}$

$z = \sqrt{2}\left[\cos\frac{2\pi}{12}(3k-1) + i\sin\frac{2\pi}{12}(3k-1)\right] = \sqrt{2}e^{i\frac{\pi}{6}(3k-1)}$

when $k=0$ $z_1 = \sqrt{2}e^{-\frac{\pi}{6}i}$

when $k=1$ $z_2 = \sqrt{2}e^{\frac{\pi}{3}i}$

when $k=2$ $z_3 = \sqrt{2}e^{\frac{5\pi}{6}i}$

when $k=3$ $z_4 = \sqrt{2}e^{\frac{8\pi}{6}i}$ -2π $z_4 = \sqrt{2}e^{-\frac{2\pi}{3}i}$

(7) (a) $r = \sqrt{(\sqrt{6})^2 + (\sqrt{2})^2} = 2\sqrt{2}$

$\theta = \arctan\left(\frac{\sqrt{2}}{\sqrt{6}}\right) = \frac{\pi}{6}$

(b) $z^{3/4} = \sqrt{6} + \sqrt{2}i$

$z^{3/4} = 2\sqrt{2}\left[\cos\left(\frac{\pi}{6} + 2k\pi\right) + i\sin\left(\frac{\pi}{6} + 2k\pi\right)\right]$

$z = (8^{\frac{1}{2}})^{4/3}\left[\cos\left(\frac{\pi + 12k\pi}{6}\right) + i\sin\left(\frac{\pi + 12k\pi}{6}\right)\right]^{4/3}$

$z = 4\left[\cos\frac{2\pi}{9}(12k+1) + i\sin\frac{2\pi}{9}(12k+1)\right] = 4e^{\frac{2\pi}{9}(12k+1)i}$

when $k=0$ $z_1 = 4e^{\frac{2\pi}{9}i}$

when $k=1$ $z_2 = 4e^{\frac{26\pi}{9}i}$ -2π. $z_2 = 4e^{\frac{8\pi}{9}i}$

when $k=2$ $z_3 = 4e^{\frac{50\pi}{9}}$ $-2\pi - 2\pi - 2\pi$ $z_3 = 4e^{-\frac{4\pi}{9}i}$