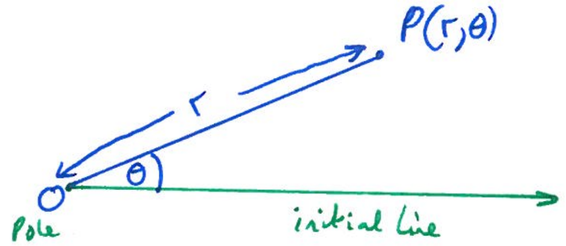
Polar Coordinates

Polar Frame of Reference

This system of reference consists of a fixed point O, called *the pole* and a line of fixed direction from O called the *initial line*.



The polar coordinates of a point P are

the distance, r, of P from O and

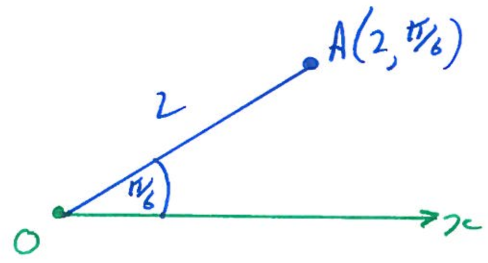
the angle, θ, between OP and the initial line.

These coordinates are written as an ordered pair (r,θ), ie (distance, angle). OP is called the *radius vector* and θ is sometimes called the *vectorial angle*.

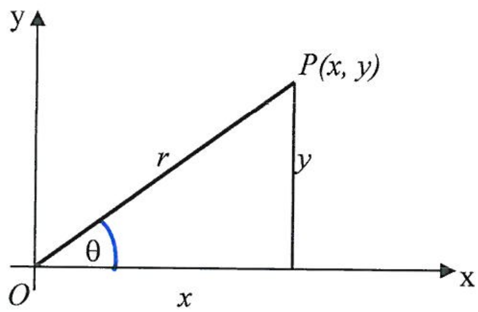
As with other circular measures, the convention that positive values of θ correspond to an anticlockwise sense of rotation from the initial line, and negative values of θ correspond to a clockwise rotation.

In FP2, we only need to consider positive values of r, corresponding to the distance $\overrightarrow{OP}$. However a negative value of r would correspond to the distance along $\overrightarrow{PO}$.

Thus the point with polar coordinates $\left(2, \frac{\pi}{6}\right)$ is the point A shown below:



Relationship between Polar and Cartesian Coordinates



If a point P has Cartesian coordinates (x, y) relative to an origin O and polar coordinates (r, θ) relative to a pole O and initial line Ox then:

$x = r\cos\theta \qquad r^2 = x^2 + y^2$

$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$

These relationships can be used to convert Cartesian coordinates into polar coordinates and polar coordinates into Cartesian coordinates.

Eg1 Given that the origin O and the pole coincide and that the initial line is Ox, find the polar coordinates of the points whose Cartesian coordinates are:
(a) (-4, 3) (b) (-1, -1) (c) (1.2, -0.9)

Eg2 Given that the pole O and the origin coincide and that the positive x-axis is the initial line, find the Cartesian coordinates of the points whose polar coordinates are:

(a) $\left(5, -\frac{\pi}{2}\right)$ (b) $\left(4, \frac{2\pi}{3}\right)$ (c) $(3, -1)$

Exercise 7A Pg130

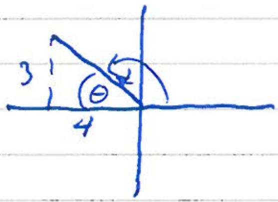
The above relationships can also be used to convert a Cartesian equation into polar form and conversely.

Eg3 Find the Cartesian equations for the curve (a) $r = 2a\cos\theta$ (b) $r = a(1 - \cos\theta)$, where a is a positive constant.

Eg4 Find the polar equations of the curve (a) $y^2 = 4x$ (b) $(x^2 + y^2)^2 = 2a^2xy$

Exercise 7B Pg 132

eg1 (a) $(-4,3)$



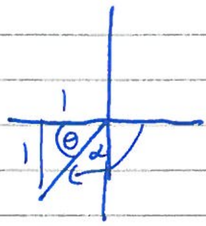
$\alpha = \pi - \tan^{-1}\left(\frac{3}{4}\right)$

$\alpha = 2.50^c$

$r = \sqrt{4^2+3^2} = 5$

$\therefore (5, 2.5^c)$

(b) $(-1,-1)$

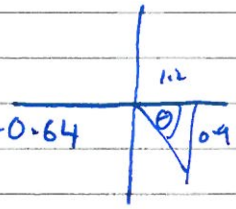


$\alpha = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

$r = \sqrt{2}$

$\left(\sqrt{2}, -\frac{3\pi}{4}\right)$

(c) $(1.2, -0.9)$



$\theta = -\tan^{-1}\left(\frac{0.9}{1.2}\right) = -0.64$

$r = \sqrt{1.2^2+0.9^2} = 1.5$

$(1.5, -0.64)$

Eg2 (a) $\left(5, -\frac{\pi}{2}\right)$

$(0, -5)$

(b) $\left(4, \frac{2\pi}{3}\right)$

$x = 4\cos\frac{2\pi}{3} = -2$

$y = 4\sin\frac{2\pi}{3} = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$(-2, 2\sqrt{3})$

(c) $(3, -1)$

$x = 3\cos(-1) = 1.62$

$y = 3\sin(-1) = -2.52$

$(1.62, -2.52)$

Eg3 (a) $r = 2a\cos\theta$

$\times r \quad r^2 = 2ar\cos\theta$ $\qquad$ but $x^2+y^2 = r^2$

$x^2+y^2 = 2ax$ $\qquad$ $x^2+y^2 = 2a.r\cos\theta$

$x^2+y^2 = 2ax$

$x^2 - 2ax + y^2 = 0$

$(x-a)^2 + y^2 - a^2 = 0$

$(x-a)^2 + y^2 = (a)^2$ i.e. circle centre $(a, 0)$ radius a

(b) $r = a(1-\cos\theta)$

$r = a - a\cos\theta$

$\times r \quad r^2 = ra - ar\cos\theta$

$x^2+y^2 = ra - ax$

$ar = x^2+y^2+ax$

$a^2r^2 = (x^2+y^2+ax)^2$

$a^2(x^2+y^2) = (x^2+y^2+ax)^2$

Eg4 (a) $y^2 = 4x$

$(r\sin\theta)^2 = 4r\cos\theta$

$r^2\sin^2\theta = 4r\cos\theta$

$r = \dfrac{4\cos\theta}{\sin^2\theta}$

(b) $(x^2+y^2)^2 = 2a^2xy$

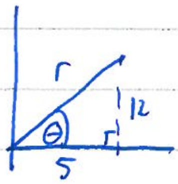
$r^4 = 2a^2r^2\sin\theta\cos\theta$

$r^2 = 2a^2\sin\theta\cos\theta$

$r^2 = a^2\sin 2\theta$

Ex 7A

① (a) $(5, 12)$

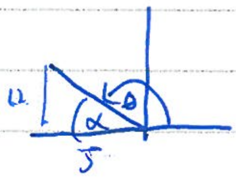


$r = \sqrt{5^2 + 12^2} = 13$

$\theta = \tan^{-1}\left(\frac{12}{5}\right) = 67.4^\circ$

$(13, 67.4)$

(b) $(-5, 12)$



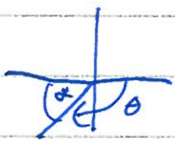
$r = 13$

$\alpha = \tan^{-1}\left(\frac{12}{5}\right) = 67.4^\circ$

$\theta = 180 - 67.4 = 112.6^\circ$

$(13, 112.6)$

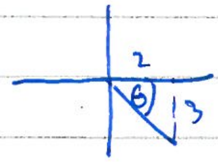
(c) $(-5, -12)$



$r = 13$, $\theta = -180 + 67.4 = -112.6$

$(13, -112.6)$

(d) $(2, -3)$

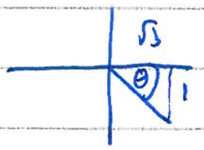


$r = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\theta = -\tan^{-1}\left(\frac{3}{2}\right) = -56.3^\circ$

$(\sqrt{13}, -56.3)$

(e) $(\sqrt{3}, -1)$

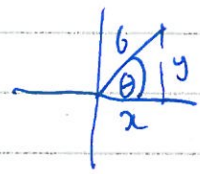


$r = \sqrt{\sqrt{3}^2 + 1^2} = 2$

$\theta = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -30^\circ$

$(2, -30)$

2 (a) $\left(6, \frac{\pi}{6}\right)$

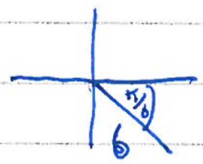


$x = r\cos\theta = 6\cos\frac{\pi}{6} = 3\sqrt{3}$

$y = r\sin\theta = 6\sin\frac{\pi}{6} = 3$

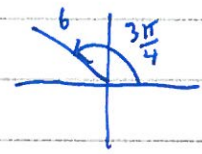
$(3\sqrt{3}, 3)$

(b) $\left(6, -\frac{\pi}{6}\right)$



$(3\sqrt{3}, -3)$

(c) $\left(6, \frac{3\pi}{4}\right)$

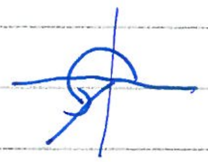


$x = 6\cos\frac{3\pi}{4} = -6\frac{\sqrt{2}}{2} = -3\sqrt{2}$

$y = 6\sin\frac{3\pi}{4} = 3\sqrt{2}$

$(-3\sqrt{2}, 3\sqrt{2})$

(d) $\left(10, \frac{5\pi}{4}\right)$



$x = 10\cos\frac{5\pi}{4} = -5\sqrt{2}$

$y = -5\sqrt{2}$

$(-5\sqrt{2}, -5\sqrt{2}$

(e) $(2, \pi)$ $x = -2$ $y = 0$

$(-2, 0)$

Ex 7B

1 (a) $r = 2 \Rightarrow r^2 = 4$ but $x^2 + y^2 = r^2$ $\therefore x^2 + y^2 = 4$

(b) $r = 3\sec\theta \Rightarrow r = \dfrac{3}{\cos\theta} \Rightarrow r\cos\theta = 3 \Rightarrow x = 3$

(c) $r = 5\operatorname{cosec}\theta \Rightarrow r = \dfrac{5}{\sin\theta} \Rightarrow r\sin\theta = 5 \Rightarrow y = 5$

2 (a) $r = 4a\tan\theta\sec\theta$

$r^2 = 16a^2\tan^2\theta\sec^2\theta$

$r^2 = 16a^2 \sec^2\theta(\sec^2\theta - 1)$

$r^2 = 16a^2(\sec^4\theta - \sec^2\theta)$

$r^2 = 16a^2\left[\dfrac{1}{\cos^4\theta} - \dfrac{1}{\cos^2\theta}\right]$

$r^2 = 16a^2\left[\dfrac{1 - \cos^2\theta}{\cos^4\theta}\right]$

$r^2\cos^4\theta = 16a^2[1 - \cos^2\theta]$

$r^2\cos^4\theta = 16a^2\sin^2\theta$

$r = 4a\tan\theta\sec\theta$

$r = 4a \cdot \dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{1}{\cos\theta}$

$r\cos^2\theta = 4a\sin\theta$

$\times r$ $\quad r^2\cos^2\theta = 4ar\sin\theta$

$(r\cos\theta)^2 = 4a(r\sin\theta)$

$x^2 = 4ay$

(b) $r = 2a\cos\theta$

$\times r$ $\quad r^2 = 2ar\cos\theta$

$x^2 + y^2 = 2ax$

(c) $r = 3a\sin\theta$

$\times r$ $\quad r^2 = 3ar\sin\theta$

$x^2 + y^2 = 3ay$

(3) (a) $r = 4(1 - \cos 2\theta)$

$r = 4\left[1 - (1 - 2\sin^2\theta)\right]$

$r = \cancel{4} \, 8\sin^2\theta$

$\times r^2$

$r^3 = 8(r\sin\theta)^2$

$(x^2+y^2)^{3/2} = 8y^2$

(b) $r = 2\cos^2\theta$

$\times r^2$

$r^3 = 2(r\cos\theta)^2$

$(x^2+y^2)^{3/2} = 2x^2$

(c) $r^2 = 1 + \tan^2\theta$

$r^2 = \sec^2\theta$

$r^2 = \dfrac{1}{\cos^2\theta}$

$(r\cos\theta)^2 = 1$

$x^2 = 1$

(4) (a) $x^2 + y^2 = 16$

$r^2 = 16$

$r = 4$

(b) $xy = 4$

$r\cos\theta \, r\sin\theta = 4$

$r^2\cos\theta\sin\theta = 4$

$\times 2 \quad r^2 \, 2\cos\theta\sin\theta = 8$

$r^2\sin 2\theta = 8$

$r^2 = \dfrac{8}{\sin 2\theta} = 8\operatorname{cosec} 2\theta$

(4) (c) $(x^2+y^2)^2 = 2xy$

$r^4 = 2r\cos\theta \, r\sin\theta$

$r^2 = \sin 2\theta$

(5) (a) $x^2+y^2-2x=0$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

(b) $(x+y)^2 = 4$

$x^2+y^2+2xy = 4$

$r^2 + 2r^2\cos\theta\sin\theta = 4$

$r^2(1+\sin 2\theta) = 4$

$r^2 = \dfrac{4}{1+\sin 2\theta}$

(c) $x-y=3$

$r\cos\theta - r\sin\theta = 3$

$r(\cos\theta - \sin\theta) = 3$

~~$r\cos(\theta+\alpha) = 3$ when $\alpha = \tan^{-1}(1) = \frac{\pi}{4}$~~

Now $a\cos\theta - b\sin\theta = R\cos(\theta+\alpha)$, where $R = \sqrt{a^2+b^2}$

$a = R\cos\alpha$

$b = R\sin\alpha$

in our case $R = \sqrt{r^2+r^2} = \sqrt{2r^2} = r\sqrt{2}$

and $r = r\sqrt{2}\cos\alpha$ — (1)

$r = r\sqrt{2}\sin\alpha$ — (2)

(2) ÷ (1) $\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$\therefore \quad r\sqrt{2}\cos\left(\theta \pm \frac{\pi}{4}\right) = 3$

$r = \dfrac{3}{\sqrt{2}}\sec\left(\theta + \dfrac{\pi}{4}\right)$

6 (a) $y = 2x$

$r\sin\theta = 2r\cos\theta$

$\tan\theta = 2$

$\theta = \arctan 2$

(b) $y = -\sqrt{3}x + a$

$r\sin\theta = -\sqrt{3}r\cos\theta + a$

$r\sin\theta + \sqrt{3}r\cos\theta = a$

$R\sin(\theta + \alpha)$ $\quad R = \sqrt{r^2 + 3r^2} = 2r$

$r = 2r\cos\alpha$ — ①

$\sqrt{3}r = 2r\sin\alpha$ — ②

① ÷ ② $\quad \sqrt{3} = \tan\alpha$

$\alpha = \frac{\pi}{3}$

$\therefore \quad 2r\sin\left(\theta + \frac{\pi}{3}\right) = a$

$r = \frac{a}{2}\operatorname{cosec}\left(\theta + \frac{\pi}{3}\right)$

(6)(c) $y = x(x-a)$

$\cancel{y = x^2 - xa}$

$\cancel{r\sin\theta = r^2\sin^2\theta}$

$r\sin\theta = r\cos\theta(r\cos\theta - a)$

$\tan\theta = r\cos\theta - a$

$r\cos\theta = \tan\theta + a$

$r = \tan\theta\sec\theta + a\sec\theta$

Polar Curve Sketching

The shape of a curve can be determined from its polar equation by listing corresponding values of θ and r and plotting these coordinates. (Values of θ are usually limited to the range $0 \le \theta \le 2\pi$.) Certain observations can reduce the amount of tabulated work however:

- If r is a function of $\cos\theta$, the curve is symmetrical about the line $\theta = 0$ since $\cos(-\theta) = \cos\theta$.
  Such a curve can be plotted by calculating values of r for angles in the range $0 \le \theta \le \pi$, drawing the curve through these points and then reflecting it in the initial line..

- If r is a function of $\sin\theta$, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$, since

  $$\sin(\pi - \theta) = \sin\pi\cos\theta - \sin\theta\cos\pi$$
  $$= 0 - (-1\sin\theta)$$
  $$= \sin\theta$$

  ie $\sin\theta = \sin(\pi - \theta)$

  eg if $\theta = \frac{2\pi}{3}$ $\quad \sin\frac{2\pi}{3} = \sin\left(\pi - \frac{2\pi}{3}\right) = \sin\frac{\pi}{3}$

  Such a curve can be plotted by plotting points calculated in the range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and reflecting this curve in the line $\theta = \frac{\pi}{2}$.

It is also helpful to establish

- What are the maximum and minimum values of r and what values of θ produce them.

- What values of θ make the trig part of the function become -1, 0 or 1.

The three most basic polar graphs are produced by the functions:

- $r = a$ this is a circle centre O and radius a

- $\theta = \alpha$ this is a half-line through O making an angle α with the initial line

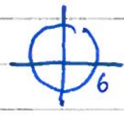
- $r = a\theta$ this is a spiral starting at O

Using Polar Graph Paper (freely available on internet) makes plotting easier too!

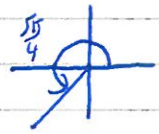
<u>Exercise 7C</u> Pg 135 (Together!)

Ex 7c

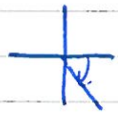
(1) (a) circle radius 6



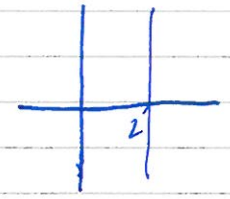
(b) $\theta = \frac{5\pi}{4}$ half line



(c) $\theta = -\frac{\pi}{4}$ half line



(2) (a) $r = 2\sec\theta$



$r = \frac{2}{\cos\theta}$

$r\cos\theta = 2$

$x = 2$

(b) $r = 3\operatorname{cosec}\theta$

$r = \frac{3}{\sin\theta}$

$r\sin\theta = 3$

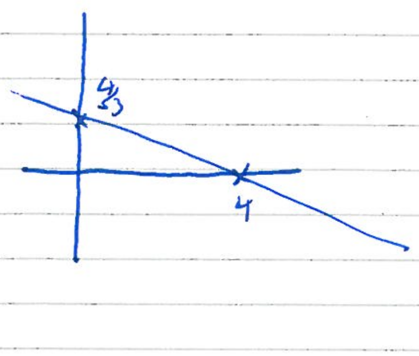
$y = 3$

(c) $r = 2\sec\left(\theta - \frac{\pi}{3}\right)$

$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$

$r\cos\theta\cos\frac{\pi}{3} + r\sin\theta\sin\frac{\pi}{3} = 2$

$\frac{r}{2}\cos\theta + \frac{\sqrt{3}r}{2}\sin\theta = 2$



$r\cos\theta + \sqrt{3}r\sin\theta = 4$

$x + \sqrt{3}y = 4$

$x = 0$ $y = \frac{4}{\sqrt{3}}$, $y = 0, x = 4$

$\left(0, \frac{4}{\sqrt{3}}\right)$ $(4, 0)$

(3) a) $r = a\sin\theta$

$\times r$

$r^2 = ar\sin\theta$

$x^2 + y^2 = ay$

$x^2 + y^2 - ay = 0$

$x^2 + \left(y - \frac{a}{2}\right)^2 - \frac{a^2}{4} = 0$

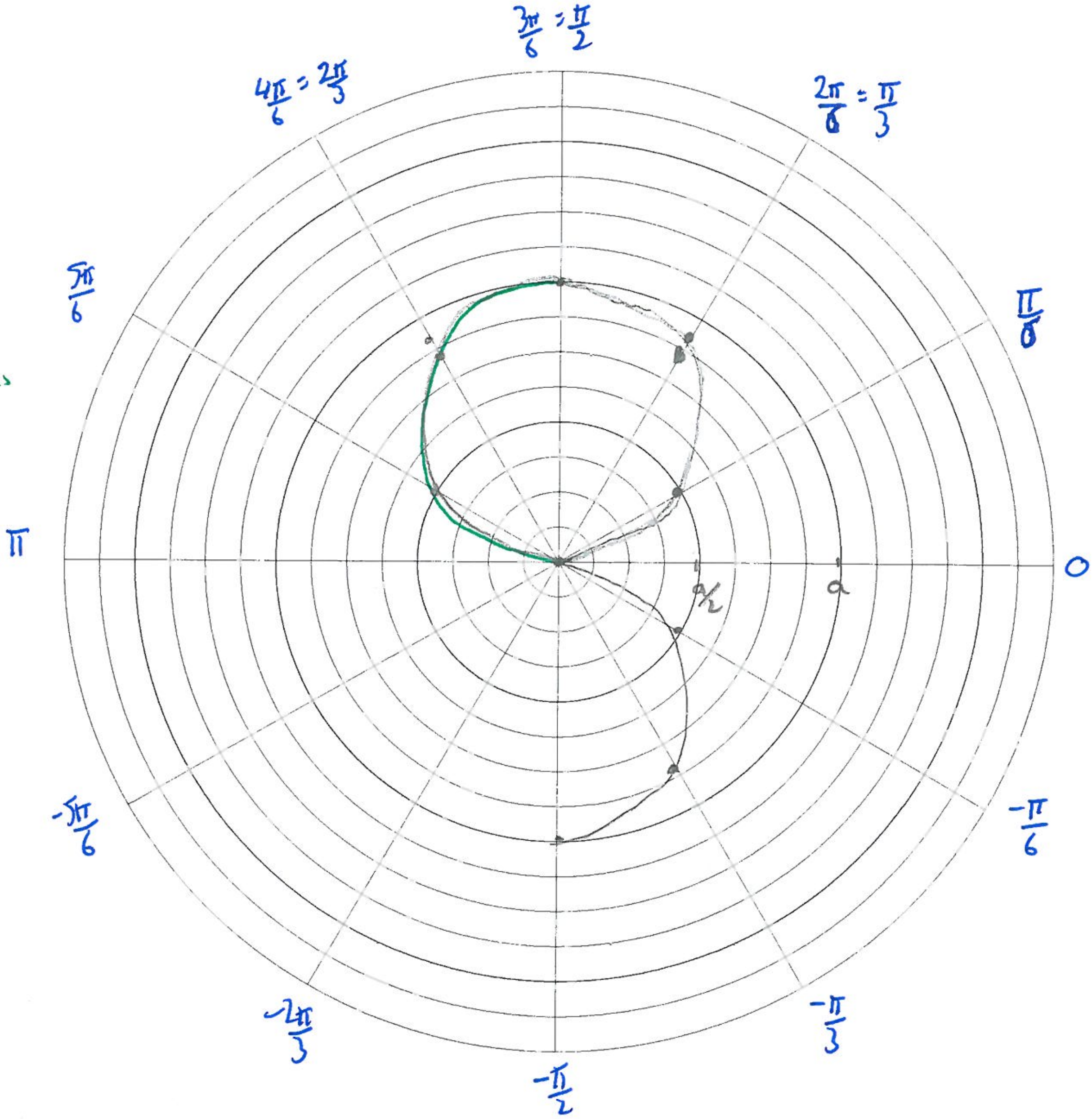
$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$ ie circle centre $\left(0, \frac{a}{2}\right)$ radius $a/2$

(b) next on graph paper

③ (a) $r = a\sin\theta$ plot $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

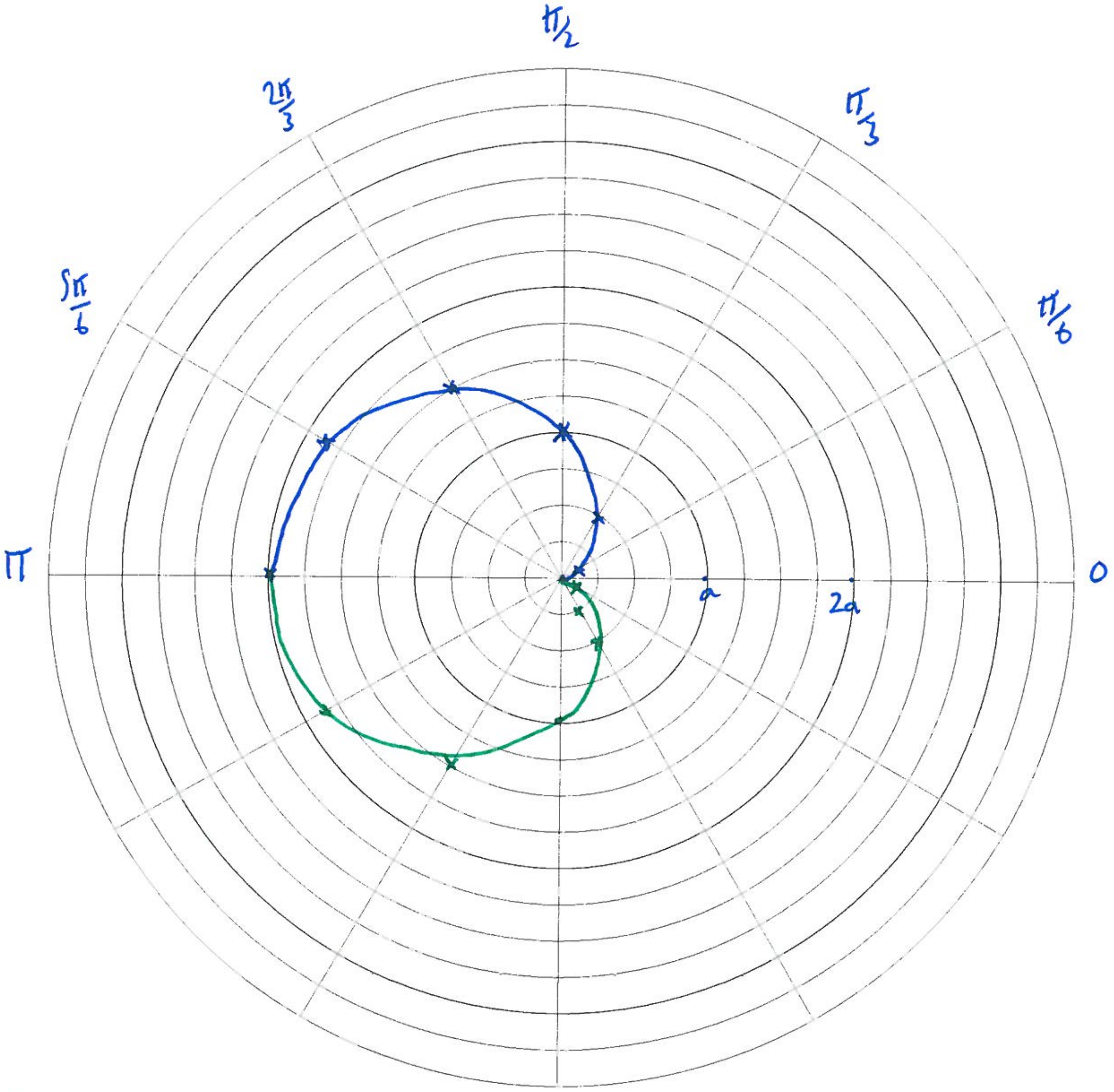
θ	$-\pi/2$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\pi/6$	$\pi/3$	$\pi/2$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
r	$-a$	$-\frac{\sqrt{3}}{2}a$	$-a/2$	0	$\frac{1}{2}a$	$\frac{\sqrt{3}}{2}a$	a

reflects



(3)(b) $r = a(1-\cos\theta)$ Cos function, so $0 \leq \theta \leq \pi$ & reflect.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\sqrt{3}/2$	0.5	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1
r	0	$0.13a$	$0.5a$	a	$1.5a$	1.9	$2a$



reflect

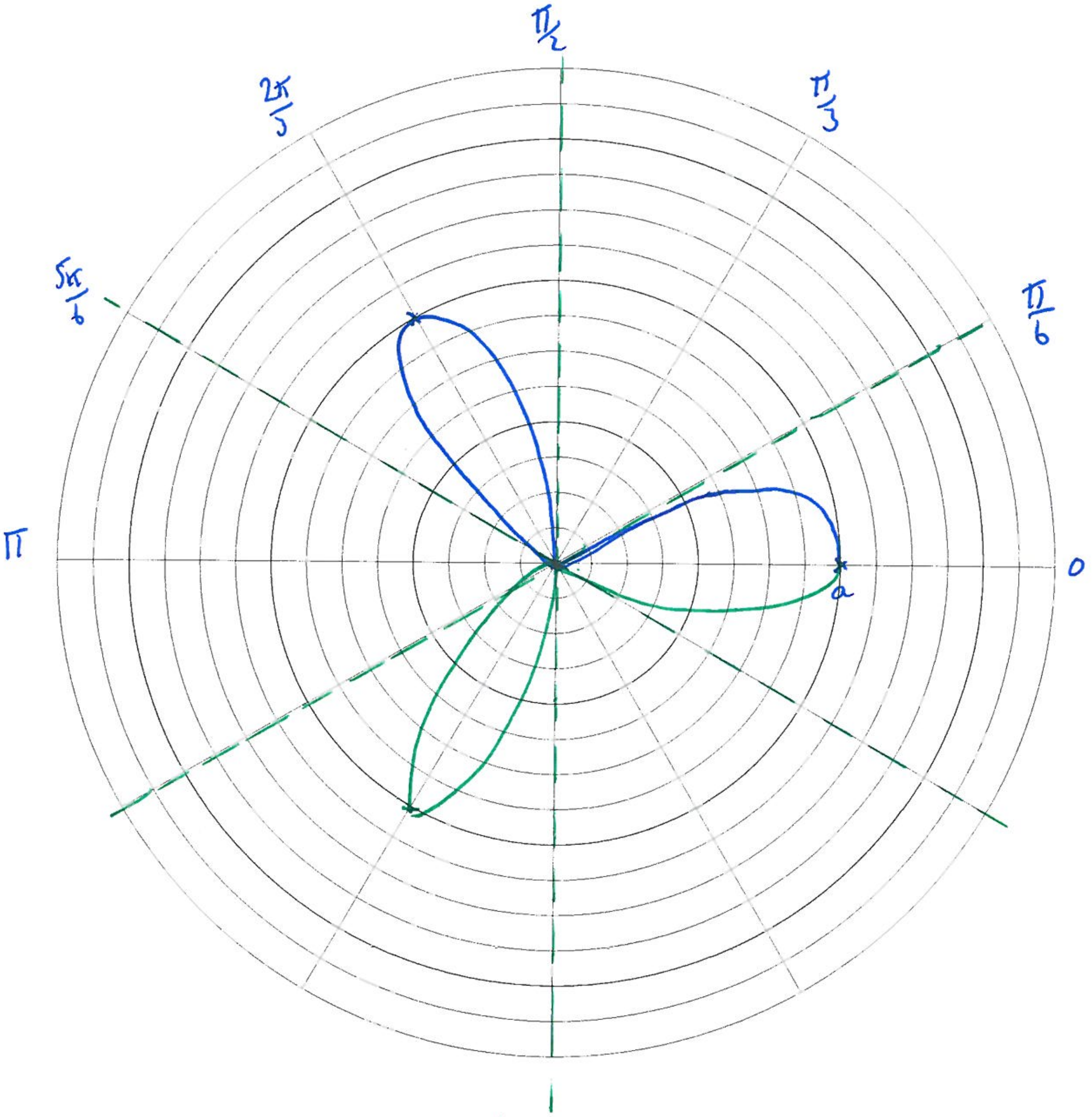
This shape known as a cardioid, comes from functions with polar equations of form $r = a(p + q\cos\theta)$

Cardioid has dimple if $q \leq p < 2q$

if $p \geq 2q$, cardioid will be egg shaped.

③ c) $r = a\cos 3\theta$

θ	0	$\frac{\pi}{6}$	$\cancel{\frac{\pi}{3}}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\cancel{\pi}$
$\cos\theta$	1	0	$\cancel{-1}$	0	1	0	$\cancel{-1}$
r	a	0	$\cancel{-a}$	0	a	0	$\cancel{-a}$



reflect

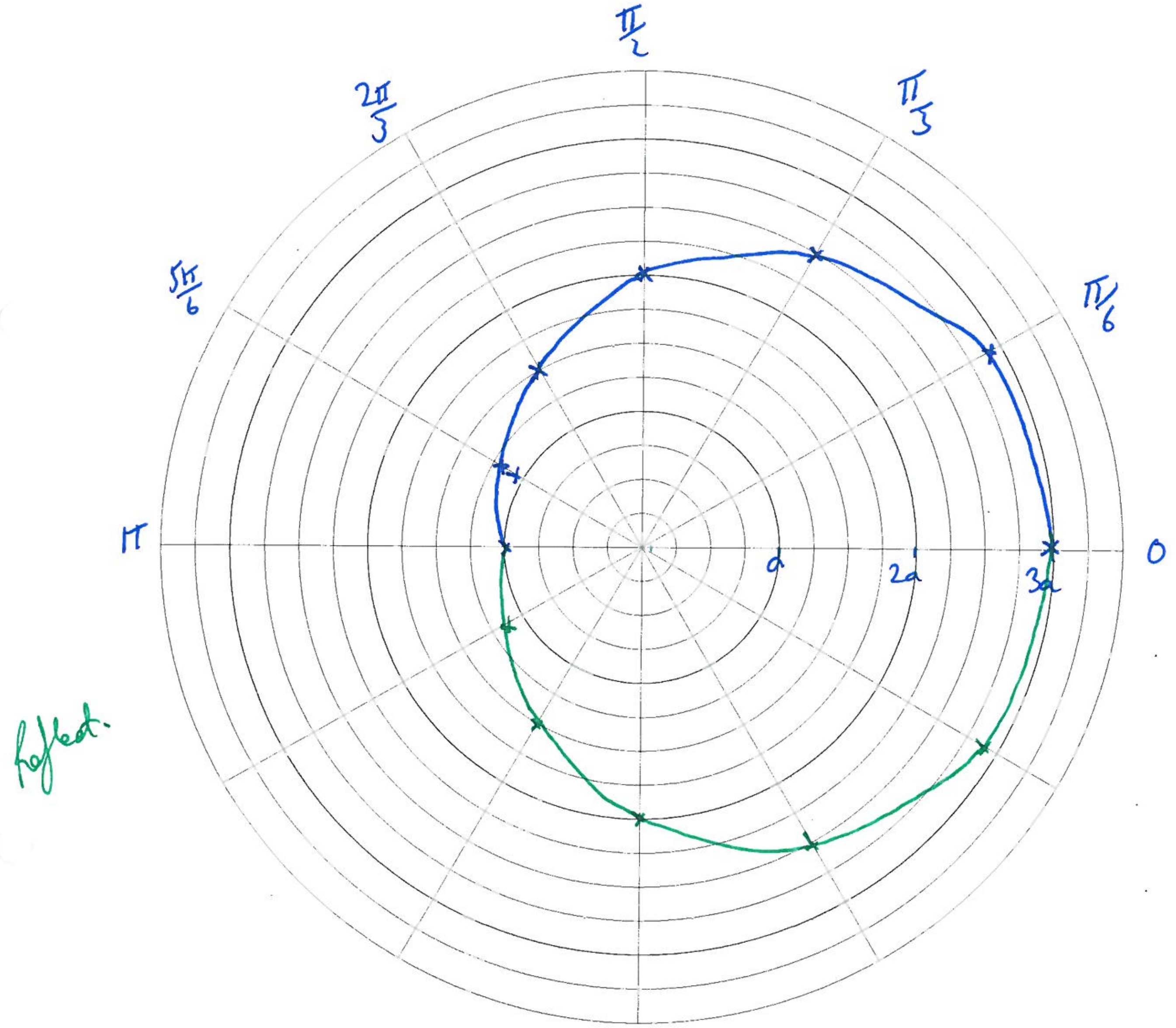
tangent to curves

Kraft

tangents to curve @ half lines $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, -\frac{\pi}{6}, -\frac{\pi}{2}, -\frac{5\pi}{6}$

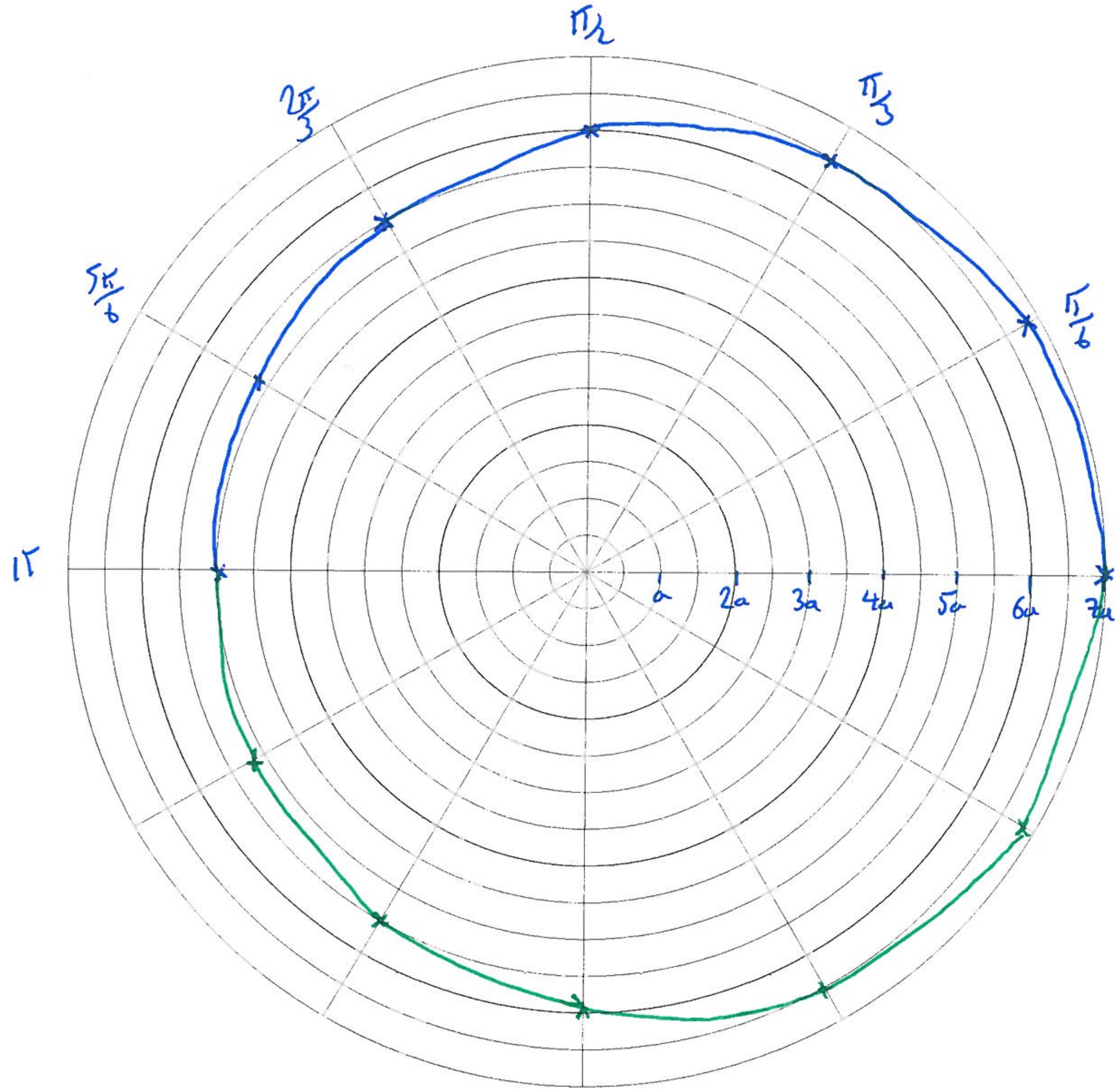
4 (a) $r = a(2+\cos\theta)$

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$\cos\theta$	1	$\sqrt{3}/2$	0.5	0	-0.5	$-\sqrt{3}/2$	-1
r	$3a$	$2.9a$	$2.5a$	$2a$	$1.5a$	$1.1a$	a



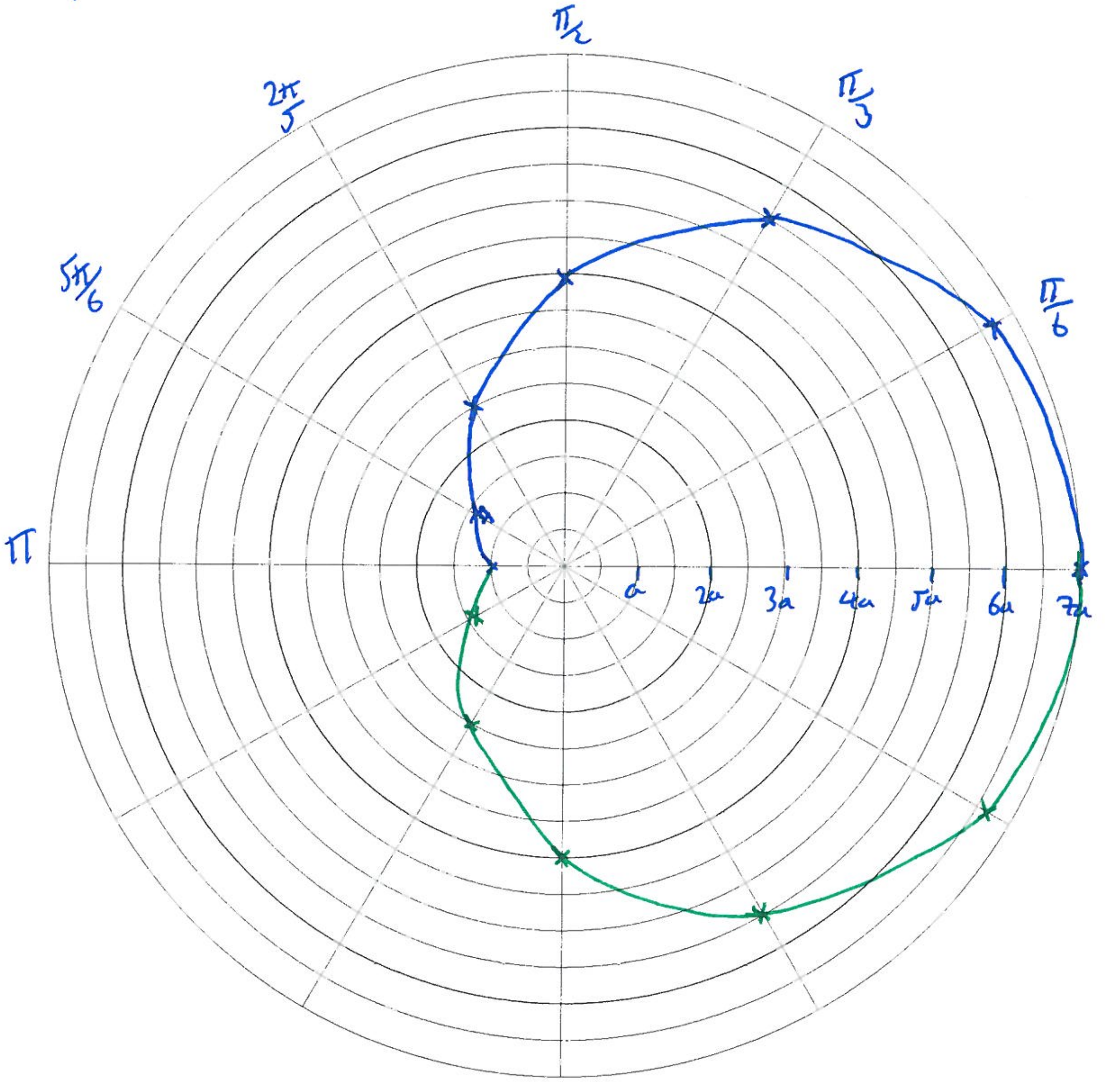
4 (b) $r = a(6 + \cos\theta)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	0.9	0.5	0	−0.5	−0.9	−1
r	7a	6.9a	6.5a	6a	5.5a	5.1a	5a



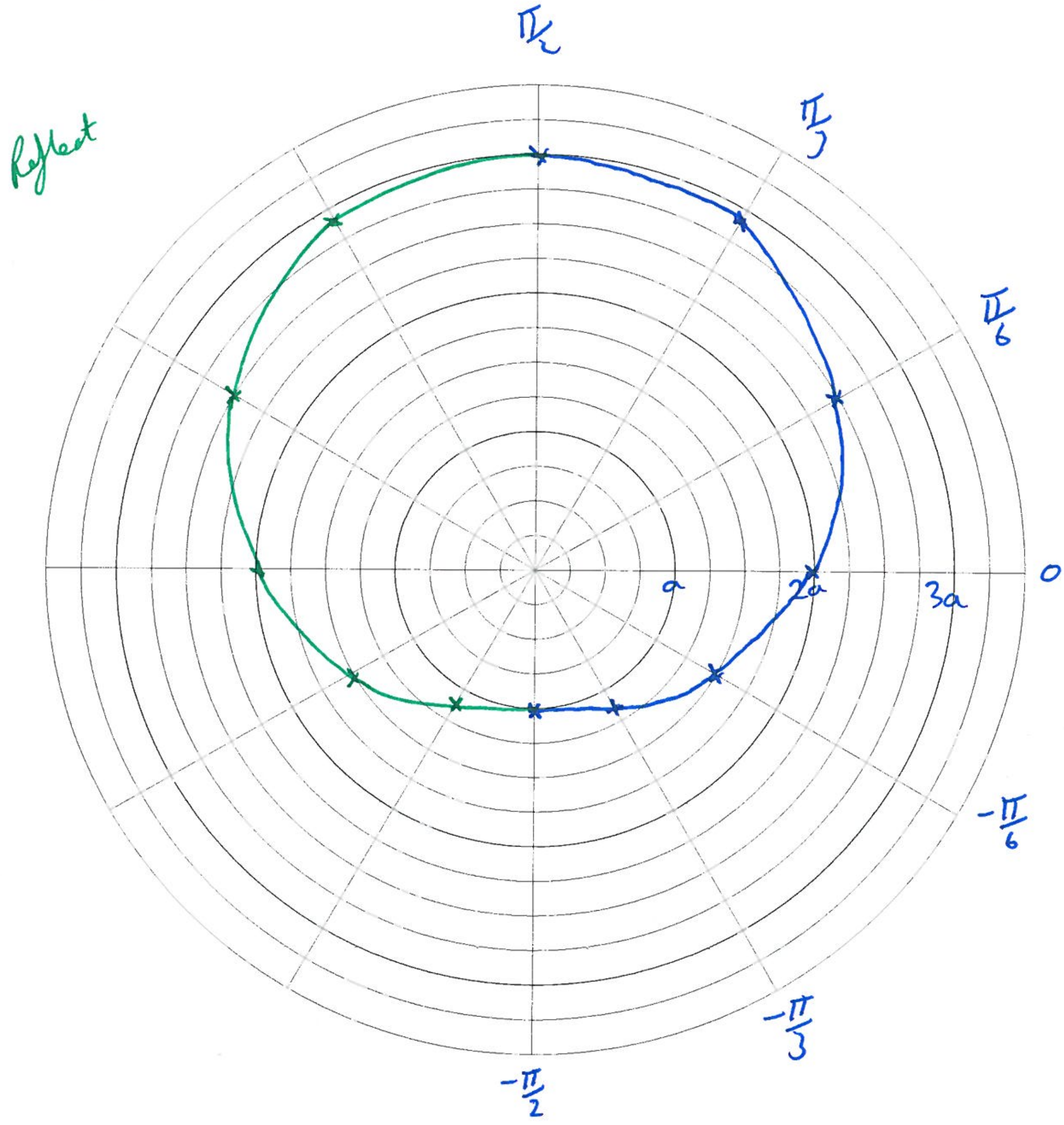
④(c) $r = a(4 + 3\cos\theta)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	0.9	0.5	0	-0.5	-0.9	-1
r	7a	6.7a	5.5a	4a	2.5a	1.3a	a



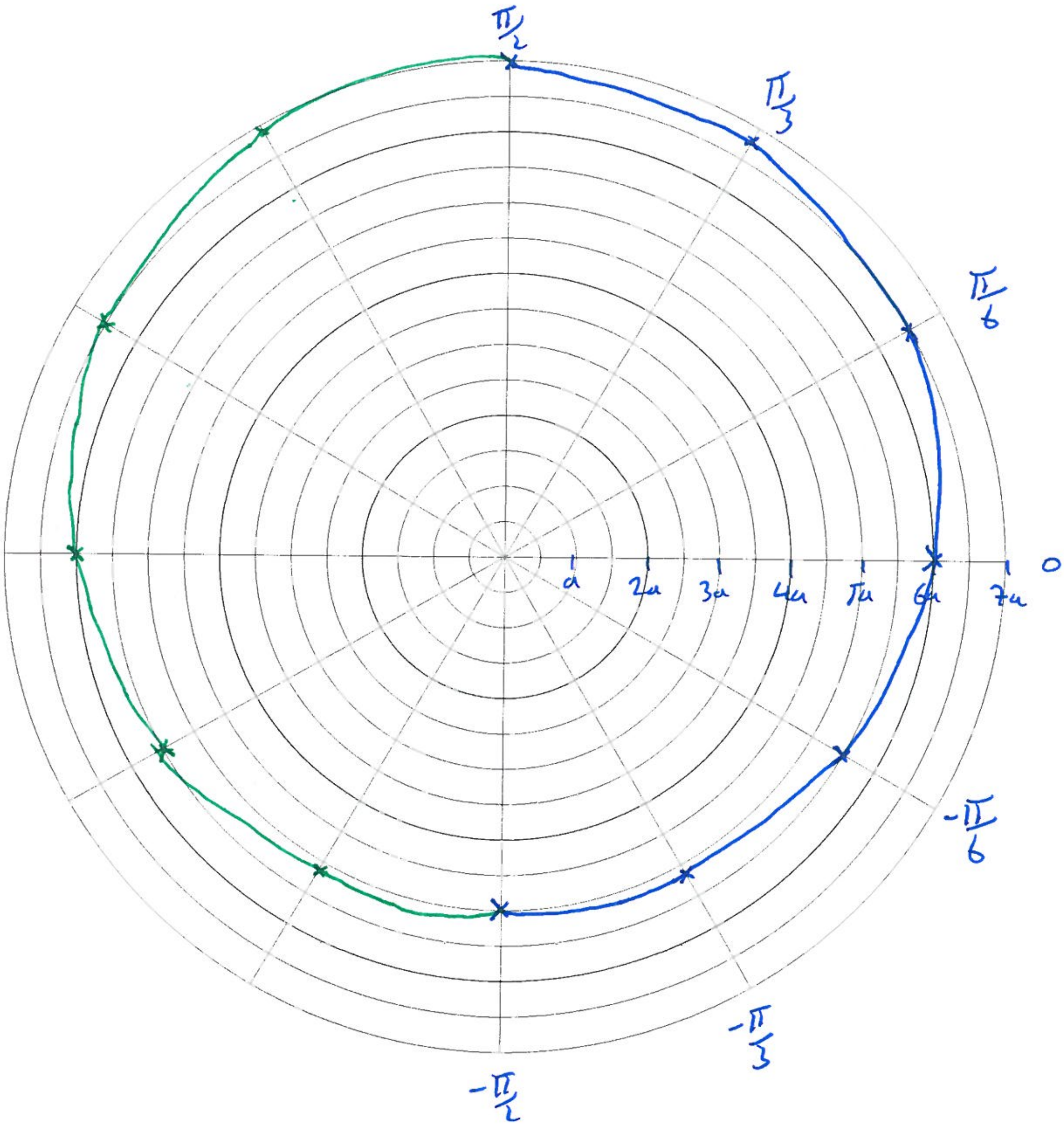
(5) (a) $r = a(2+\sin\theta)$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	-0.9	-0.5	0	0.5	0.9	1
r	a	$1.1a$	$1.5a$	$2a$	$2.5a$	$2.9a$	$3a$



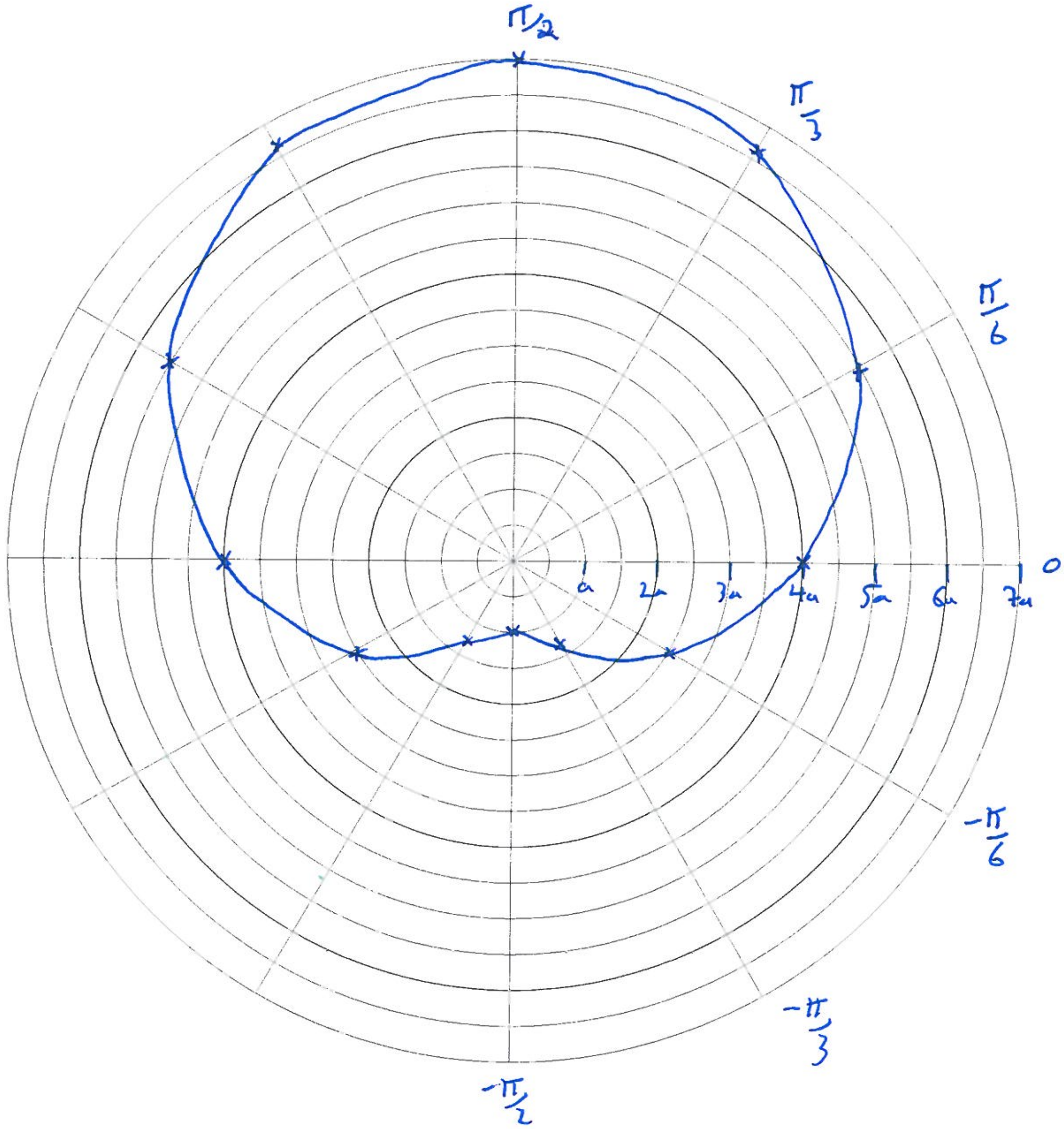
(5) 1b) $r = a(6 + \sin\theta)$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	-0.9	-0.5	0	0.5	0.9	1
r	$5a$	$5.1a$	$5.5a$	$6a$	$6.5a$	$6.9a$	$7a$



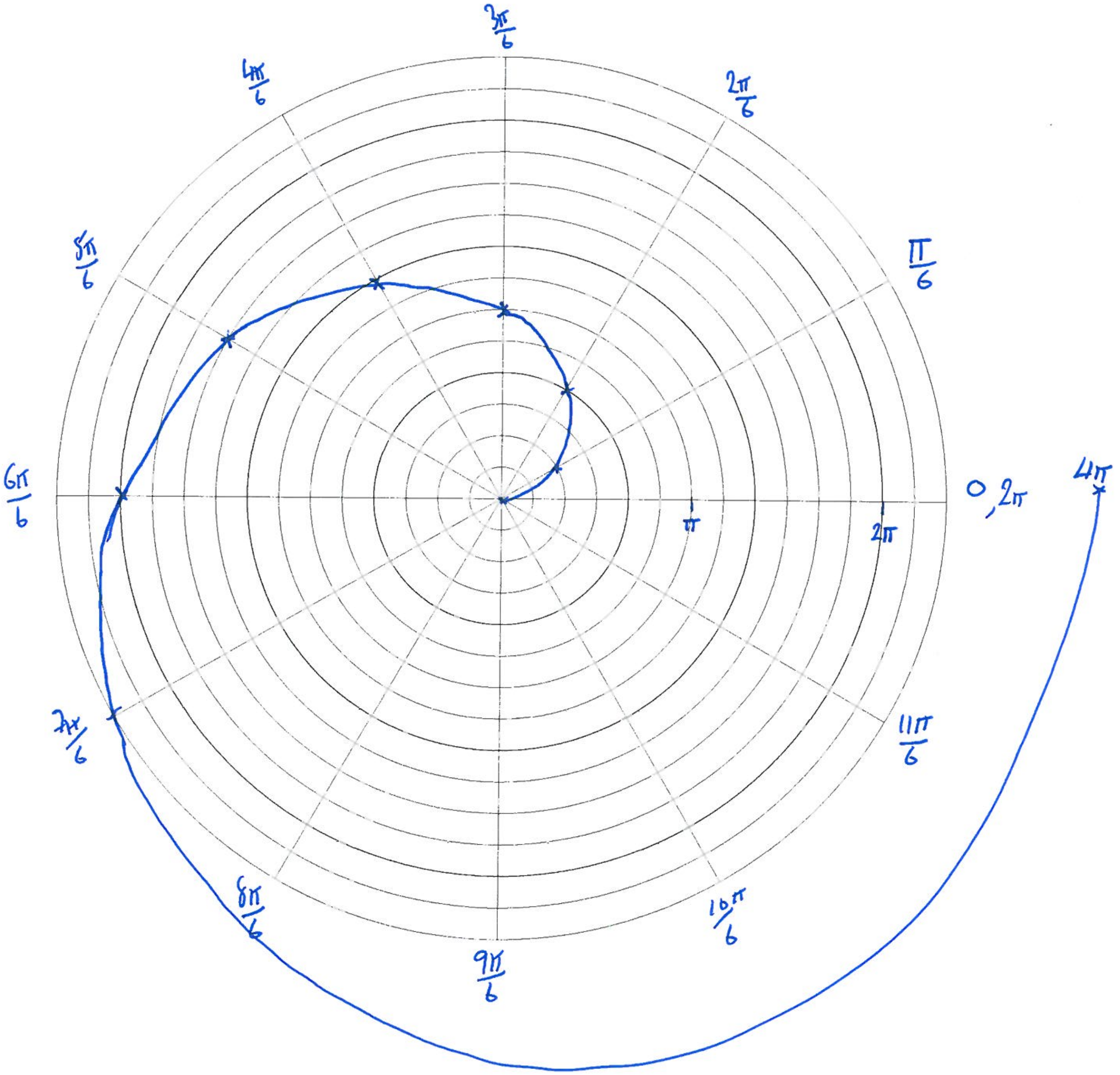
5 (c) $r = a(4 + 3\sin\theta)$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	-0.9	-0.5	0	0.5	0.9	1
r	a	$1.3a$	$2.5a$	$4a$	$5.5a$	$6.7a$	$7a$



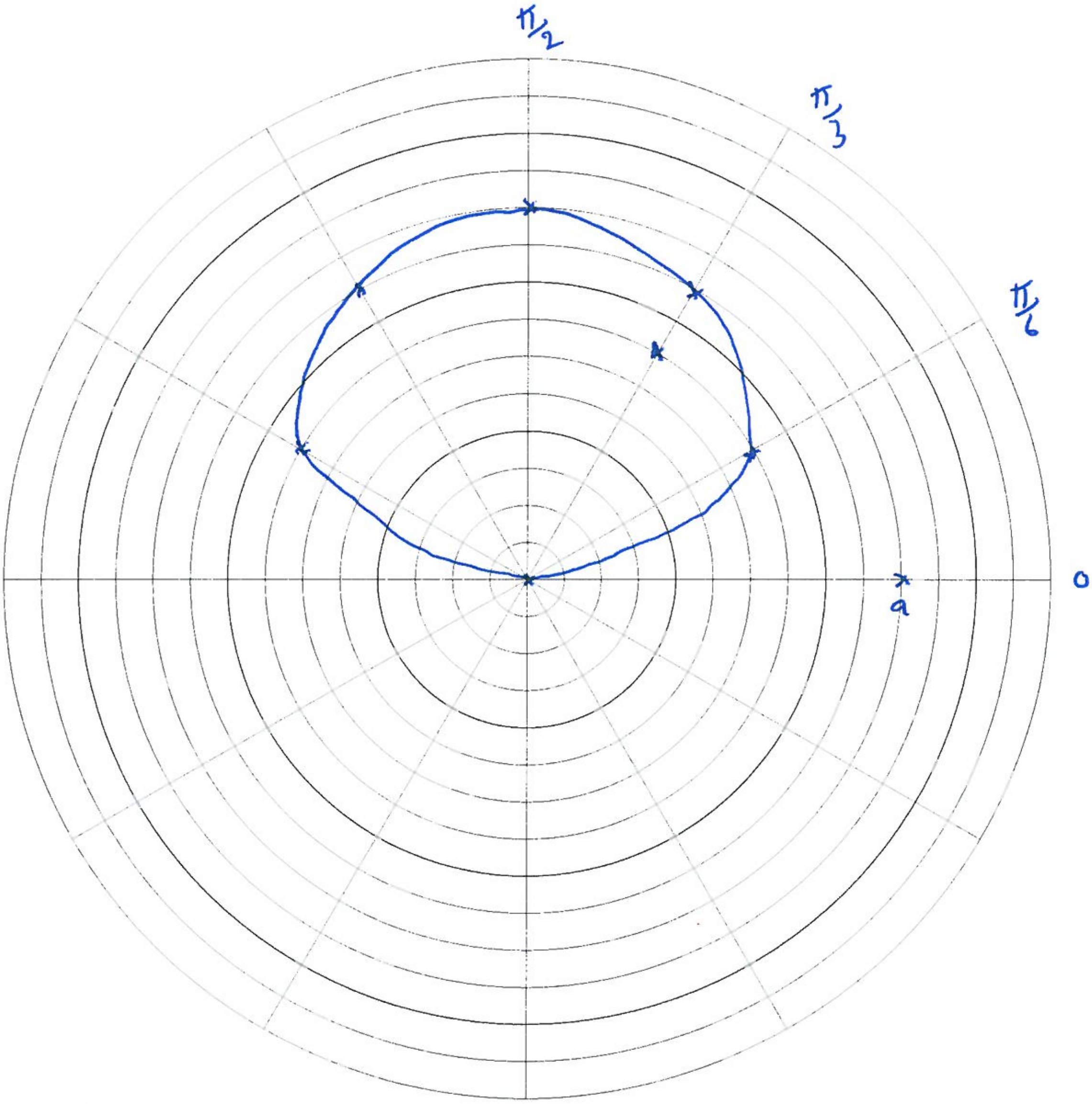
6 (a) $r = 2\theta$

θ	0	$\pi/6$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
r	0	$\frac{2\pi}{6}$	$\frac{4\pi}{6}$	$\frac{6\pi}{6}$	$\frac{8\pi}{6}$	$\frac{10\pi}{6}$	$\frac{12\pi}{6}$



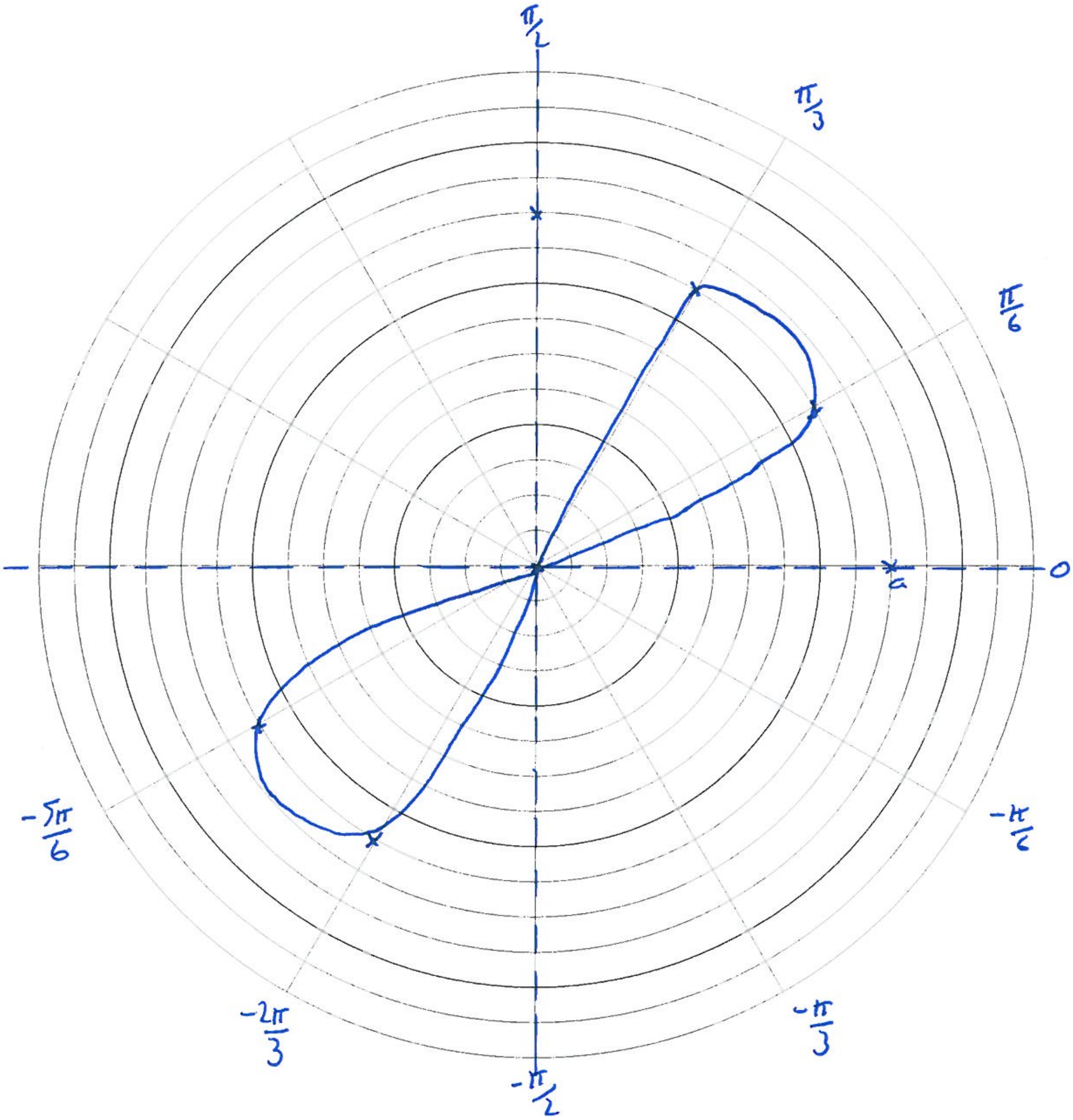
(6)(b) $r^2 = a^2 \sin\theta$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	-0.9	-0.5	0	0.5	0.9	1
r	$\times$	$\times$	$\times$	0	$\frac{a}{\sqrt{2}}$ $0.7a$	$0.9a$	a



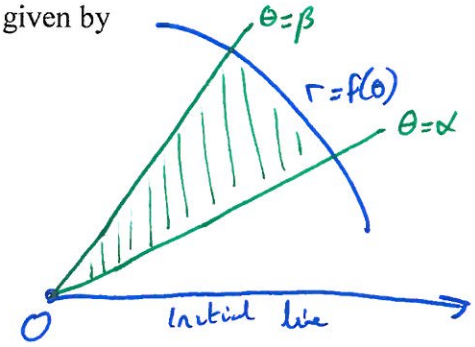
(6) (c) $r^2 = a^2 \sin 2\theta$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$-\frac{2\pi}{3}$	$-\frac{5\pi}{6}$
r	0	x	x	0	0.9a	0.9a	0	0.9a	0.9a



Areas of Regions Expressed in Polar Coordinates

For the curve with polar equation $r = f(\theta)$, the area bounded by the curve and the half-lines $\theta = \alpha$ and $\theta = \beta$ is given by



$$Area = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Eg5 Find the area of the cardioid with polar equation $r = 2a(1+\cos\theta)$

Eg6 Find the area of the finite region bounded by the half lines $\theta = 0$ and $\theta = \frac{\pi}{3}$ and the curve with polar equation $r = a(1+\tan\theta)$, where a is a positive constant.

Ex7D Page 139 (these get nasty!!)

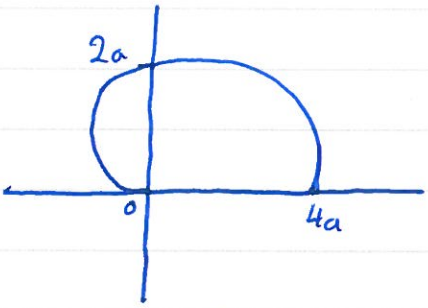
eg5 $\quad r = 2a(1+\cos\theta)$

$\theta = 0 \quad r = 4a$

$\theta = \frac{\pi}{2} \quad r = 2a$

$\theta = \pi \quad r = 0$

Sketch



Curve symmetrical about initial line $\therefore$ area of cardioid is twice
Area between $\theta = 0$ & $\theta = \pi$

$$\text{Area} = 2 \times \frac{1}{2}\int_0^{\pi} 4a^2(1+\cos\theta)\,d\theta$$

$$= 4a^2\int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta\,d\theta$$

$$= 4a^2\int_0^{\pi} 1 + 2\cos\theta + \left[\frac{1}{2} + \frac{\cos 2\theta}{2}\right] d\theta$$

$$= 4a^2\int_0^{\pi} \frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\,d\theta$$

$$= 4a^2\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi}$$

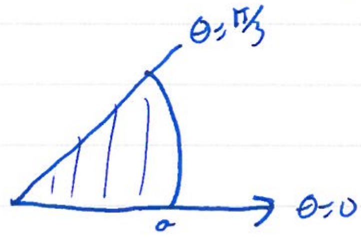
$$= 4a^2\left[\left(\frac{3\pi}{2} + 0 + \frac{1}{4}0\right) - 0\right]$$

$$= 6\pi a^2$$

Eg6 $\quad r = a(1+\tan\theta)$

$\theta = 0 \quad r = a$

$\theta = \frac{\pi}{3} \quad r = a(1+\sqrt{3})$



$$A = \frac{1}{2}\int_0^{\pi/3} a^2(1+\tan\theta)^2 \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/3} 1 + 2\tan\theta + \tan^2\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/3} 2\tan\theta + (1+\tan^2\theta) \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/3} 2\tan\theta + \sec^2\theta \, d\theta$$

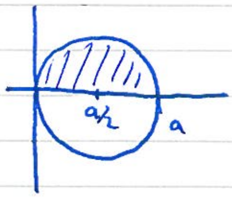
$$= \frac{a^2}{2}\Big[2\ln(\sec\theta) + \tan\theta\Big]_0^{\pi/3}$$

$$= \frac{a^2}{2}\Big[2\ln 2 + \sqrt{3} - 0 - 0\Big]$$

$$= \frac{a^2}{2}\Big[2\ln 2 + \sqrt{3}\Big].$$

Ex 7D

(1) $$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} a^2 \cos^2\theta \, d\theta$$



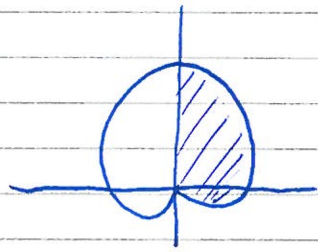
$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{\cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{4}\int_0^{\frac{\pi}{2}} 1 + \cos 2\theta \, d\theta$$

$$= \frac{a^2}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{4}\left[\frac{\pi}{2}\right] = \frac{a^2\pi}{8}$$

(2) $$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2(1+\sin\theta)^2 \, d\theta$$



$$= \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + 2\sin\theta + \sin^2\theta \, d\theta$$

$$= \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + 2\sin\theta + [1 - \cos^2\theta] \, d\theta$$

$$= \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + 2\sin\theta + 1 - \frac{1}{2} - \frac{\cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{2} + 2\sin\theta - \frac{\cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{4}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} 3 + 2\sin\theta - \cos 2\theta \, d\theta$$

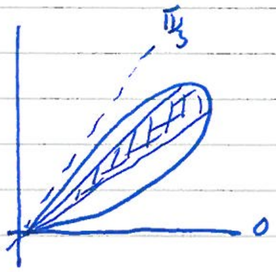
$$= \frac{a^2}{4}\left[3\theta - 2\cos\theta - \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{a^2}{4}\left[\left(\frac{3\pi}{2} - 0 - 0\right) - \left(-\frac{3\pi}{2} - 0 - 0\right)\right] = \frac{3\pi a^2}{4}$$

(3) $$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} a^2(\sin 3\theta)^2 \, d\theta$$

one loop between $0 \leq \theta \leq \frac{\pi}{3}$

$\sin(3\times 0) = 0$ $\quad \sin\left(3\times\frac{\pi}{3}\right) = 0$



$$= \frac{a^2}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 1 - \cos^2 3\theta \, d\theta$$

$$= \frac{a^2}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 1 - \frac{1}{2} - \frac{\cos 6\theta}{2} \, d\theta$$

$$= \frac{a^2}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 1 - \cos 6\theta \, d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{a^2}{4}\left[\left(\frac{\pi}{4} - -\frac{1}{6}\right) - \left(\frac{\pi}{6} - 0\right)\right]$$

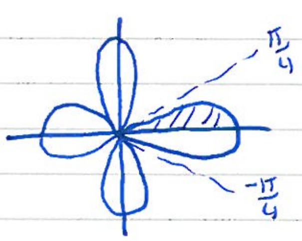
$$= \frac{a^2}{4}\left[\frac{\pi}{12} + \frac{1}{6}\right]$$

$$= \frac{a^2}{48}\left[\pi + 2\right]$$

(4). $$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} a^2 \cos 2\theta \, d\theta$$

tangents @ $\theta = \frac{\pi}{4}$ $\quad \cos\left(2\times\frac{\pi}{4}\right) = 0$

$\theta = -\frac{\pi}{4}$ $\quad \cos\left(2\times -\frac{\pi}{4}\right) = 0$

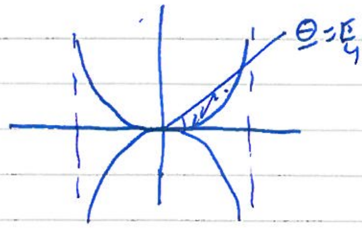


$$A = \frac{a^2}{2}\left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$A = \frac{a^2}{4}\left[1\right]$$

$$= \frac{a^2}{4}$$

(5) $r^2 = a^2 \tan\theta$



$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} a^2 \tan\theta \, d\theta$$

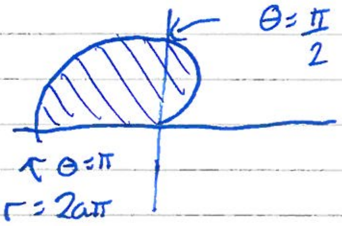
$$A = \frac{a^2}{2}\int_0^{\frac{\pi}{4}} \tan\theta \, d\theta.$$

$$A = \frac{a^2}{2}\Big[\ln(\sec x)\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{a^2}{2}\Big[\ln\sqrt{2} - \ln 1\Big]$$

$$= \frac{a^2}{2}\cdot\frac{1}{2}\ln 2 = \frac{a^2 \ln 2}{4}$$

(6) $r = 2a\theta$

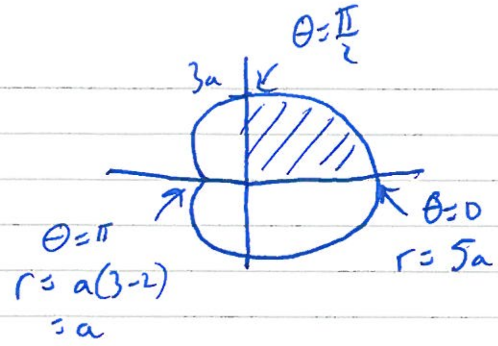


$r = 2a \times \frac{\pi}{2} = a\pi$

$$A = \frac{1}{2}\int_0^{\pi} 4a^2\theta^2 \, d\theta$$

$$A = 2a^2\left[\frac{\theta^3}{3}\right]_0^{\pi} = \frac{2a^2\pi^3}{3}$$

⑦ $r = a(3 + 2\cos\theta)$



$$A = \frac{1}{2}\int_0^{\pi/2} a^2(3+2\cos\theta)^2\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} 9 + 12\cos\theta + 4\cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} 9 + 12\cos\theta + 4\left[\frac{1}{2} + \frac{\cos 2\theta}{2}\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} 9 + 12\cos\theta + 2 + 2\cos 2\theta\,d\theta$$

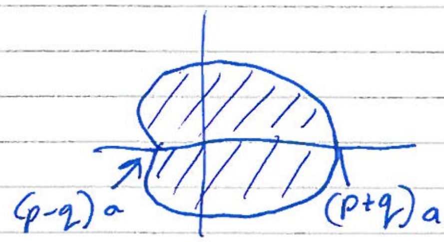
$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} 11 + 12\cos\theta + 2\cos 2\theta\,d\theta$$

$$= \frac{a^2}{2}\left[11\theta + 12\sin\theta + \sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\frac{11\pi}{2} + 12 + 0 - 0 - 0 - 0\right]$$

$$= \frac{a^2}{4}\left[11\pi + 24\right].$$

⑧ $r = a(p + q\cos\theta)$



$$\text{Shaded Area} = \frac{2 \times 1}{2}\int_0^{\pi} a^2(p+q\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi} p^2 + 2pq\cos\theta + q^2\cos^2\theta\,d\theta$$

$$= a^2\int_0^{\pi} p^2 + 2pq\cos\theta + q^2\left[\frac{1}{2} + \frac{\cos 2\theta}{2}\right] d\theta$$

(8) contd

$$= a^2 \int_0^{\pi} p^2 + \frac{q^2}{2} + 2pq\cos\theta + \frac{q^2}{2}\cos 2\theta \, d\theta$$

$$= a^2 \left[p^2\theta + \frac{q^2}{2}\theta + 2pq\sin\theta + \frac{q^2}{4}\sin 2\theta \right]_0^{\pi}$$

$$= a^2 \left[p^2\pi + q^2\frac{\pi}{2} + 2pq\times 0 + 0 \right]$$

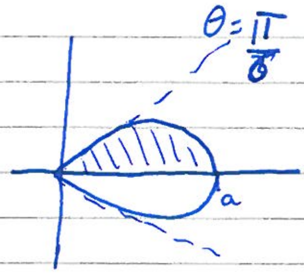
$$= \cancel{\frac{a^2}{2}\left[2p^2\pi + 4pq + q^2 \right]}$$

$$= \frac{a^2}{2}\left[p^2\pi + q^2\frac{\pi}{2} \right]$$

$$= \frac{a^2\pi}{4}\left[2p^2 + q^2 \right]$$ As required.

(9). $r = a\cos 3\theta$ tangent when $\theta = \frac{\pi}{6}$

$\cos 3\frac{\pi}{6} = 0$



Area of shaded $= \frac{1}{2}\int_0^{\pi/6} a^2\cos^2 3\theta \, d\theta$

$$= \frac{a^2}{2}\int_0^{\pi/6} \left(\frac{1}{2} + \frac{\cos 6\theta}{2} \right) d\theta$$

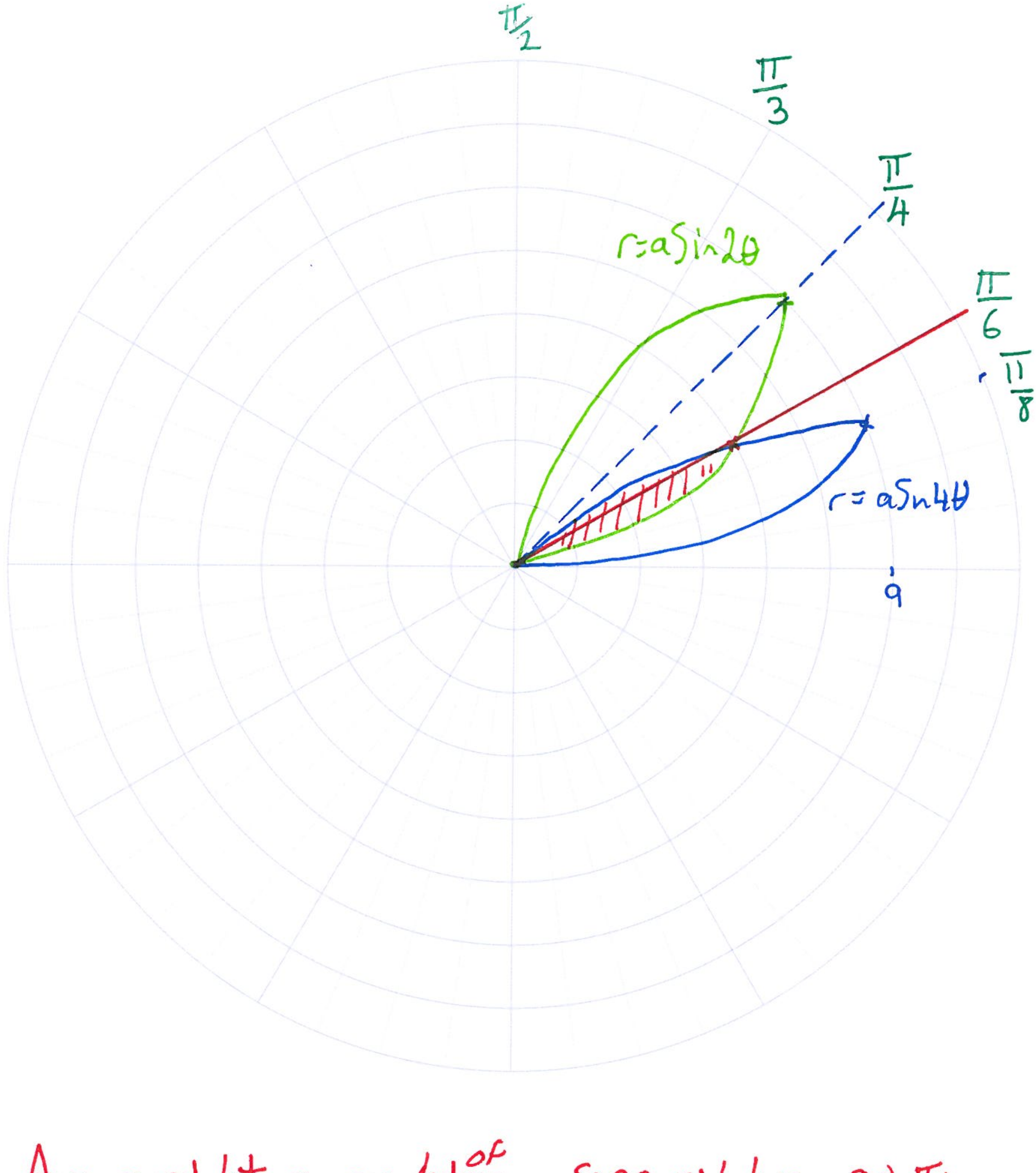
$$= \frac{a^2}{4}\int_0^{\pi/6} 1 + \cos 6\theta \, d\theta$$

$$= \frac{a^2}{4}\left[\theta + \frac{1}{6}\sin 6\theta \right]_0^{\pi/6}$$

$$= \frac{a^2}{4}\left[\frac{\pi}{6} + 0 \right] = \frac{a^2\pi}{24}$$

$\therefore$ area of one loop $= 2 \times \frac{a^2\pi}{24} = \frac{a^2\pi}{12}$

Q10.



Area = red bit = area ~~between~~ of $a\sin 2\theta$ ~~and~~ between 0 & $\pi/6$

+ area of $a\sin 4\theta$ between $\frac{\pi}{6}$ and $\frac{\pi}{4}$

(10) $r = a\sin 4\theta$

$\theta = 0 \quad r = 0$

$\theta = \frac{\pi}{8} \quad r = a$

$\theta = \frac{\pi}{4} \quad r = 0$

$r = a\sin 2\theta$

$\theta = 0 \quad r = 0$

$\theta = \frac{\pi}{4} \quad r = a$

$\theta = \frac{\pi}{2} \quad r = 0$

Curves intersect when $a\sin 4\theta = a\sin 2\theta$

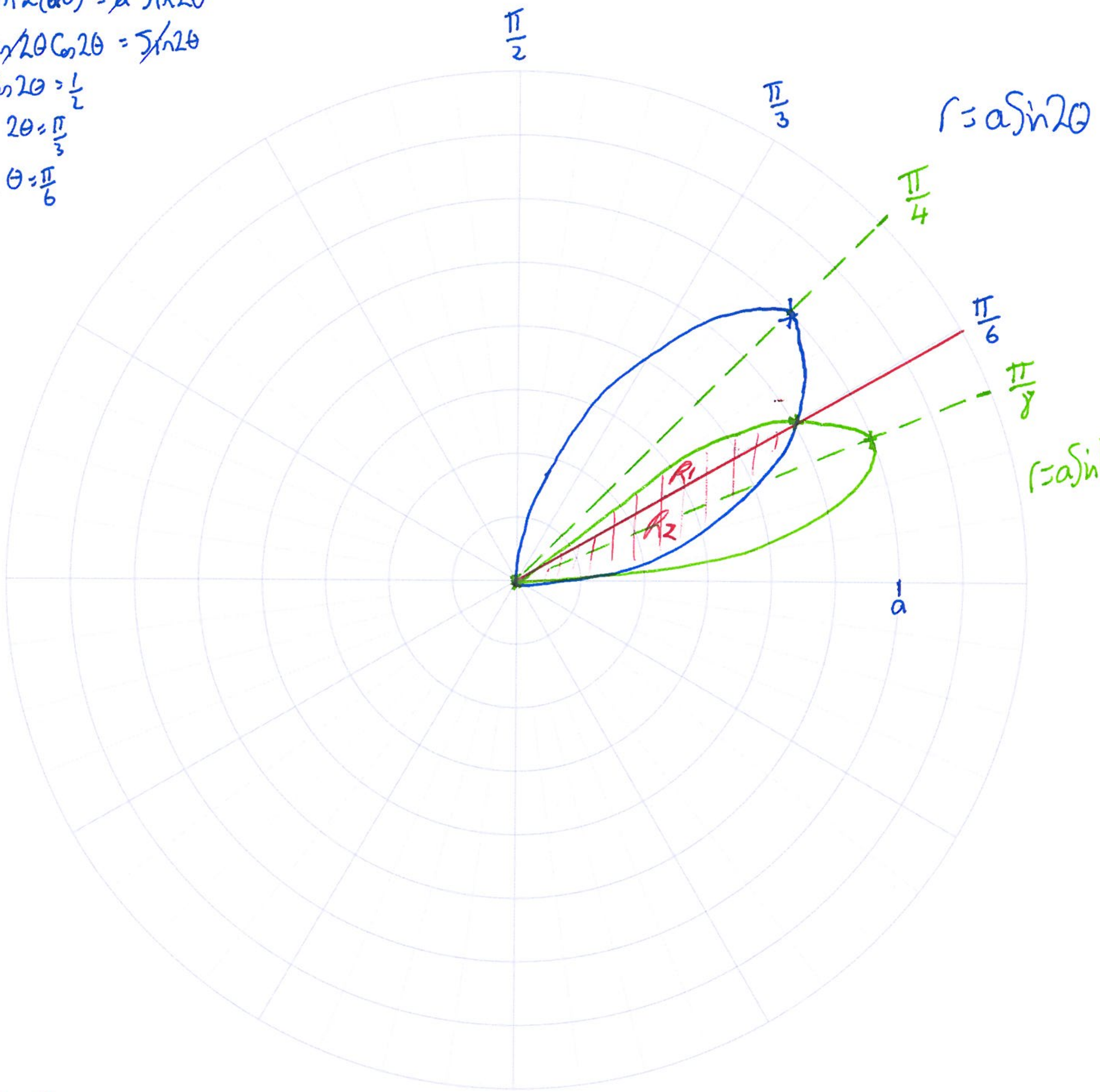
$\cancel{a}\sin 2(2\theta) = \cancel{a}\sin 2\theta$

$2\sin 2\theta\cos 2\theta = \sin 2\theta$

$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{6}$



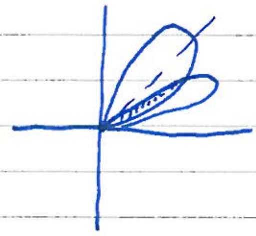
Area of R_1 = area of $r = a\sin 4\theta$ between $\frac{\pi}{6}$ & $\frac{\pi}{4}$

Area of R_2 = area of $r = a\sin 2\theta$ between 0 & $\frac{\pi}{6}$

We need shaded area $= R_1 + R_2$

10a) $r = a\sin 4\theta$ one loop between $0 \le \theta \le \frac{\pi}{4}$

$r = a\sin 2\theta$ one loop between $0 \le \theta \le \frac{\pi}{2}$



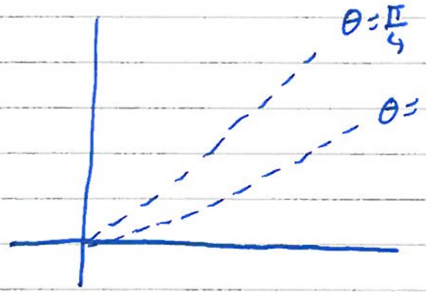
Intersect when $a\sin 2\theta = a\sin 4\theta$

$$\sin 2\theta = 2\cancel{\sin 2\theta}\cos 2\theta$$

$$\cos 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{3}$$

$$\theta = \frac{\pi}{6}$$



See polar graph paper

Area below black line $= \frac{1}{2}\int_0^{\frac{\pi}{6}} a^2\sin^2 2\theta \, d\theta$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{6}} \frac{1}{2} - \frac{\cos 4\theta}{2} \, d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{a^2}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$$

Area above black line $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} a^2\sin^2 4\theta \, d\theta$

$$= \frac{a^2}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2} - \frac{\cos 8\theta}{2} \, d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{1}{8}\sin 8\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{a^2}{4}\left[\frac{\pi}{4} - 0 - \frac{\pi}{6} - \frac{\sqrt{3}}{16}\right] = \frac{a^2}{4}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{16}\right]$$

$\therefore$ Area of shaded region $= \frac{a^2}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{8} + \frac{\pi}{12} - \frac{\sqrt{3}}{16}\right] = \frac{a^2}{4}\left[\frac{\pi}{4} - \frac{3\sqrt{3}}{16}\right]$

Q11. $r = 1 + \sin\theta$

$\theta = 0 \quad r = 1$

$\theta = \frac{\pi}{2} \quad r = 2$

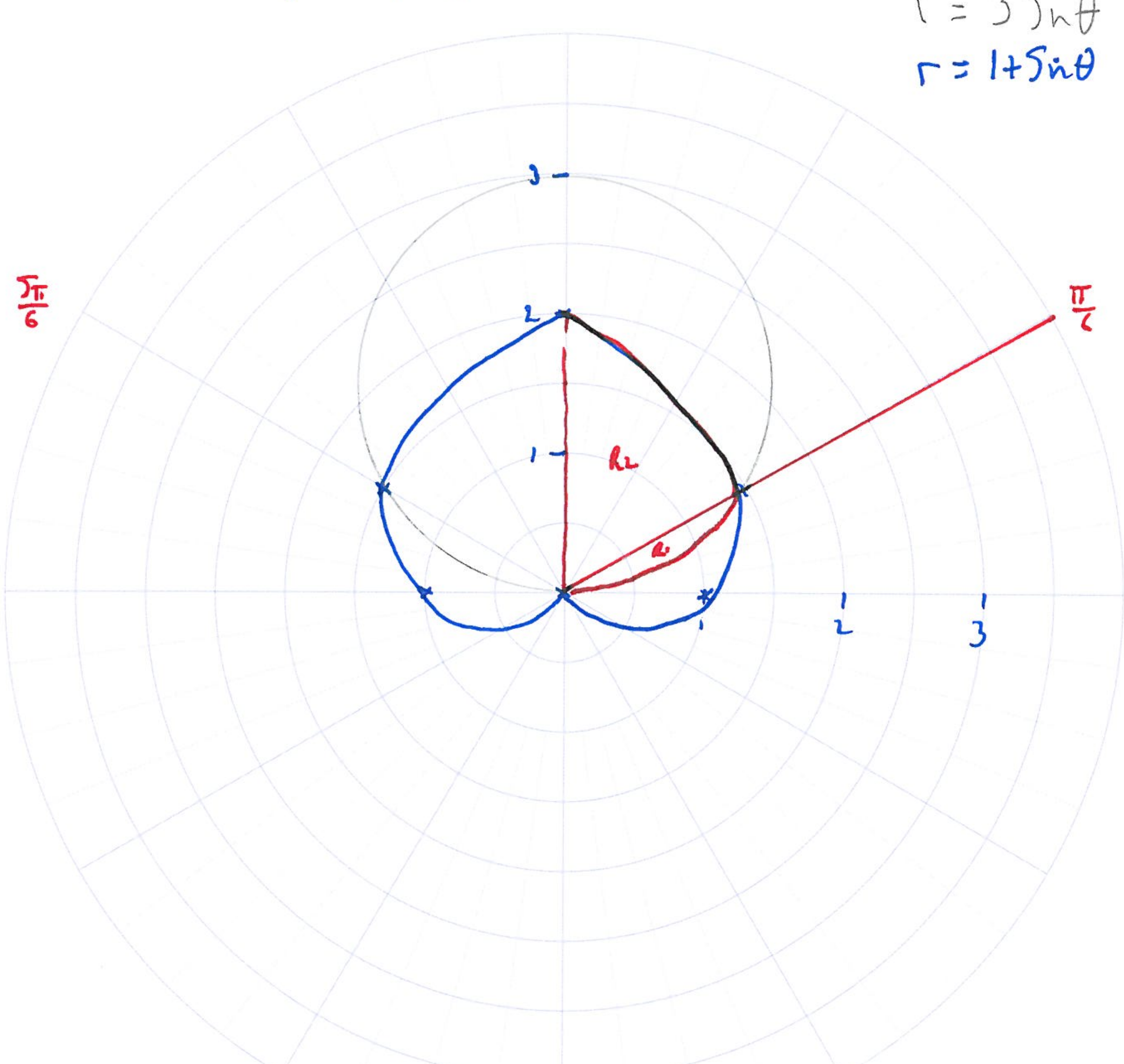
$\theta = \pi \quad r = 1$

$\theta = \frac{3\pi}{2} \quad r = 0$

$r = 3\sin\theta \Rightarrow$ circle centre $(0, 1.5)$ rad 1.5

Intersect when $1 + \sin\theta = 3\sin\theta$

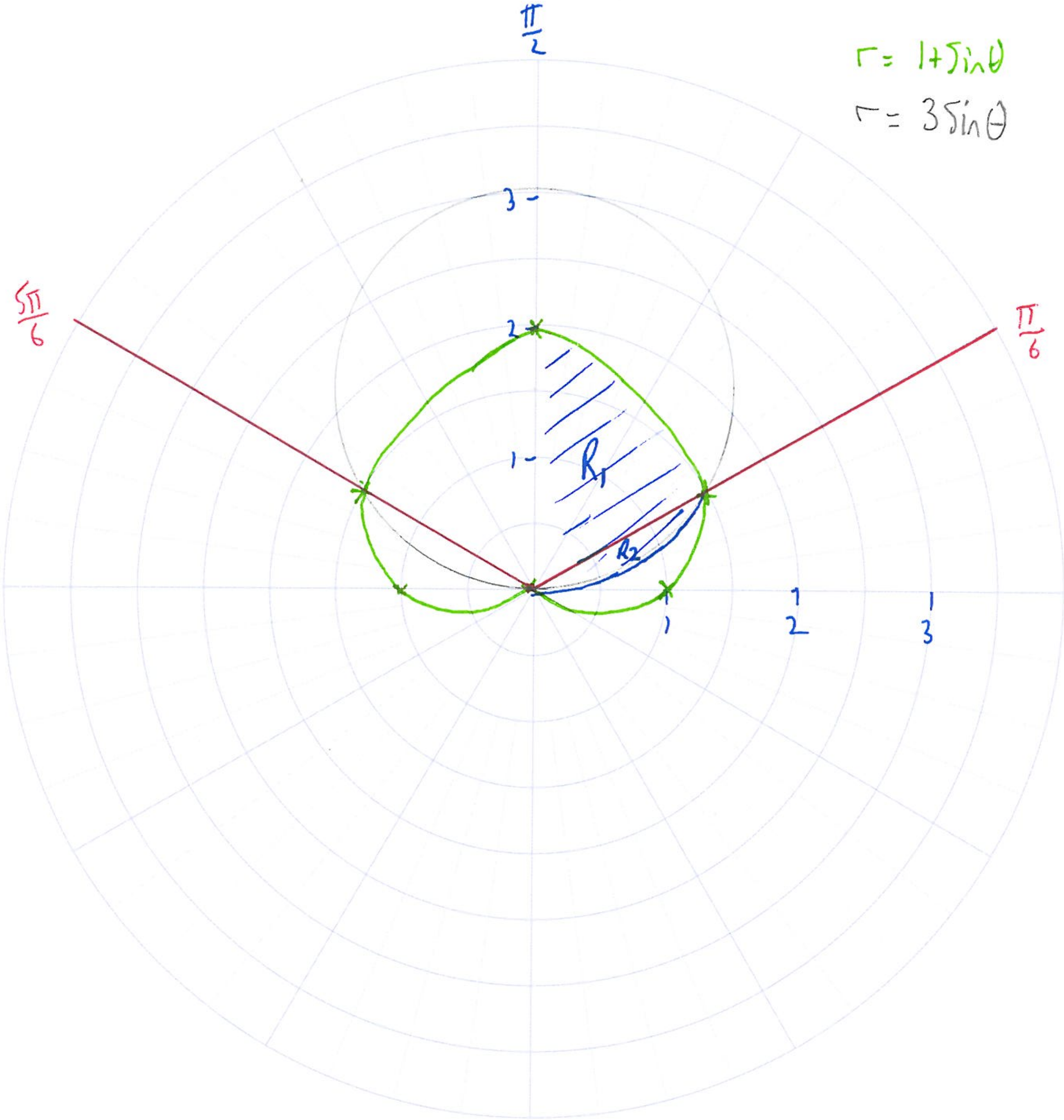
$\sin\theta = \frac{1}{2} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$



$R_1 =$ area of $3\sin\theta$ between 0 & $\frac{\pi}{6}$

$R_2 =$ area of cardiod between $\frac{\pi}{6}$ & $\frac{\pi}{2}$

(11)



Need R_1 = area of $1+\sin\theta$ between $\frac{\pi}{6}$ and $\frac{\pi}{2}$

R_2 = area of $3\sin\theta$ between 0 and $\frac{\pi}{6}$

and enclosed area $= 2[R_1 + R_2]$.

Q11 contd from graph paper

$$R_1 = \frac{1}{2}\int_0^{\pi/6} 9\sin^2\theta \, d\theta$$

$$= \frac{9}{2}\int_0^{\pi/6} \frac{1}{2} - \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{9}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$$

$$= \frac{9}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] = \frac{3\pi}{8} - \frac{9\sqrt{3}}{16}$$

$$R_2 = \frac{1}{2}\int_{\pi/6}^{\pi/2} 1 + 2\sin\theta + \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/2} 1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left[\frac{3\theta}{2} - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{3\pi}{4} - 0 - 0 - \left(\frac{1\pi}{4} - \sqrt{3} - \frac{\sqrt{3}}{8}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{9\sqrt{3}}{8}\right] = \frac{\pi}{4} + \frac{9\sqrt{3}}{16}$$

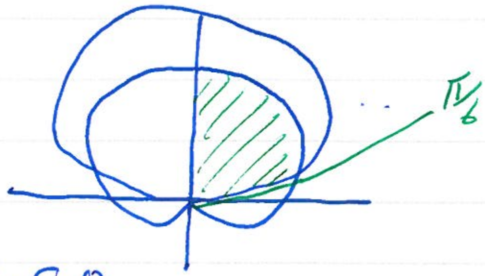
$$\text{Area} = 2\left[R_1 + R_2\right]$$

$$= 2\left[\frac{3\pi}{8} - \cancel{\frac{9\sqrt{3}}{16}} + \frac{\pi}{4} + \cancel{\frac{9\sqrt{3}}{16}}\right]$$

$$= \frac{5\pi}{4}.$$

(11) $r = 3\sin\theta$ circle centre $(0, \frac{3}{2})$ radius $\frac{3}{2}$

$r = 1 + \sin\theta$

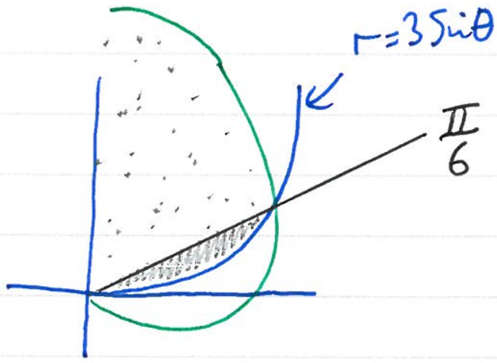


Intersect when $3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$



Area of shaded bit $= \frac{1}{2}\int_0^{\frac{\pi}{6}} 9\sin^2\theta \, d\theta$

$$= \frac{9}{2}\int_0^{\frac{\pi}{6}} \frac{1}{2} - \frac{\cos 2\theta}{2} \, d\theta = \frac{9}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{9}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] = \frac{3\pi}{8} - \frac{9\sqrt{3}}{16}$$

Area of dotted bit $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \sin\theta)^2 \, d\theta$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 + 2\sin\theta + \sin^2\theta \, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 + 2\sin\theta + \frac{1}{2} - \frac{\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3}{2} + 2\sin\theta - \frac{\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{2}\left[\frac{3\theta}{2} - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

(11) contd $= \frac{1}{2}\left[\left(\frac{3\pi}{4} - 0\right) - \left(\frac{\pi}{4} - 2\times\frac{\sqrt{3}}{2} - \frac{1}{4}\times\frac{\sqrt{3}}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{3\pi}{4} - \frac{\pi}{4} + \sqrt{3} + \frac{\sqrt{3}}{8}\right]$

$= \frac{\pi}{4} + \frac{9\sqrt{3}}{16}$

Since Region is symmetrical $\therefore$ total Area $= 2\left[\frac{\pi}{4} + \frac{9\sqrt{3}}{16}\right] + 2\left[\frac{3\pi}{8} - \frac{9\sqrt{3}}{16}\right]$

$= \frac{5\pi}{4}$

Eg 7 $\qquad r = 2a(1+\cos\theta)$ — (1)

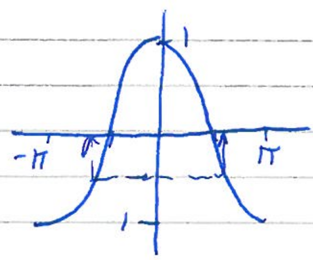
(a) for tangents ~~parallel~~ perpendicular to initial line $\frac{dx}{d\theta} = 0$

$$x = r\cos\theta$$

$$x = 2a(\cos\theta + \cos^2\theta)$$

$$\frac{dx}{d\theta} = -2a\sin\theta - 4a\cos\theta\sin\theta = 0$$

$$2a\sin\theta\left[1 + 2\cos\theta\right] = 0$$



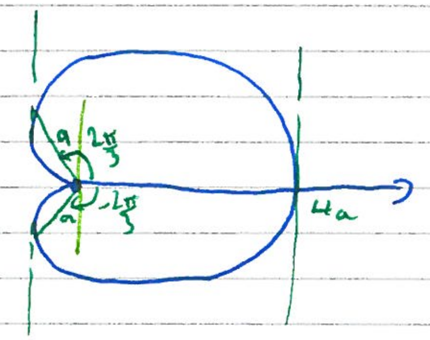
$\sin\theta = 0 \qquad$ or $\qquad \cos\theta = -\frac{1}{2}$

$\theta = 0, \pi \qquad\qquad \theta = \pm\frac{2\pi}{3}$

in (1) $\quad r = 2a(1+\cos 0) = 4a \qquad r = 2a\left(1+\cos\left(\frac{2\pi}{3}\right)\right) = a$

$\therefore$ perpendicular tangent @ $(4a, 0)$, $\left(a, \frac{2\pi}{3}\right)$, $\left(a, -\frac{2\pi}{3}\right)$, $(0, \pi)$

tangent inside curve.



(b) for tangents parallel to initial line, $\frac{dy}{d\theta} = 0$

$$y = r\sin\theta$$

$$y = 2a\sin\theta + 2a\cos\theta\sin\theta$$

$$y = 2a\sin\theta + a\sin 2\theta$$

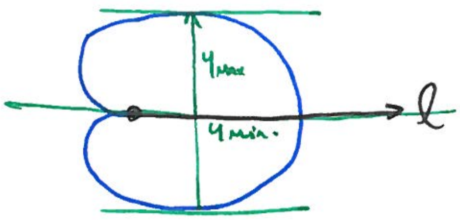
$$\frac{dy}{d\theta} = 2a\cos\theta + 2a\cos 2\theta = 0$$

$$\cos 2\theta = -\cos\theta$$

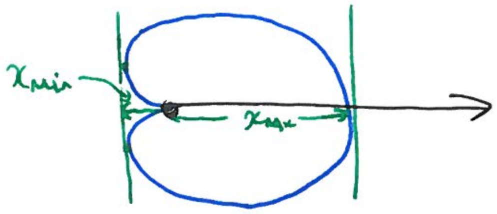
$$2\cos^2\theta + \cos\theta - 1 = 0$$

Tangents parallel to and at right angles to the initial line l..

Tangents to a polar curve that are parallel to the initial line l will occur at positions where y takes a maximum or minimum value, ie when $r\sin\theta$ takes a maximum or minimum value. This occurs when $\frac{dy}{d\theta}=0$



Similarly, tangents to a polar curve that are perpendicular to the initial line l occur at positions where x takes a maximum or minimum value, ie when $r\cos\theta$ takes a maximum or minimum value. This occurs when $\frac{dx}{d\theta}=0$



Eg7 Find, for the cardioid with equation $r=2a(1+\cos\theta)$, those points on the curve where

(a) tangents to the curve are perpendicular to l
(b) tangents to the curve are parallel to l

Find also the equations of these parallel tangents

Exercise 7E Page 143

FP2 All Complete!!

Ex 7E

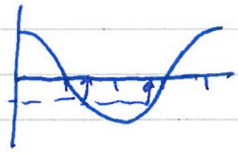
(1) $r = a(1+\cos\theta)$ — ①

For perp to initial line set $\frac{dx}{d\theta} = 0$

Now $x = r\cos\theta = a(\cos\theta + \cos^2\theta)$

$\frac{dx}{d\theta} = -\sin\theta + 2\cos\theta . -\sin\theta = 0$

$\sin\theta(2\cos\theta + 1) = 0$



either $\sin\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\theta = 0, \pi$ $\qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

in ① $r = a(1+1)$
$= 2a$ $\qquad r = \frac{a}{2}$

$\therefore$ Tangents perpendicular to initial line @ $(0, 2a), (\frac{a}{2}, \frac{2\pi}{3}), (\frac{a}{2}, \frac{4\pi}{3})$

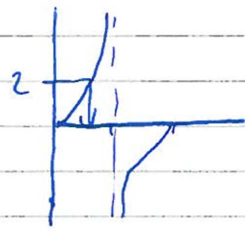
(2) $r = e^{2\theta}$ — ①

(a) for perp to initial line set $\frac{dx}{d\theta} = 0$

$x = e^{2\theta}\cos\theta$

$\frac{dx}{d\theta} = e^{2\theta} . -\sin\theta + \cos\theta . 2e^{2\theta}$

$e^{2\theta}(2\cos\theta - \sin\theta) = 0$



either $e^{2\theta} = 0$ or $2\cos\theta - \sin\theta = 0$

$2\cos\theta = \sin\theta$

$2 = \tan\theta$

$\theta = 1.107^c$

in ① $r = e^{2(1.107)} = 9.15$

$\therefore$ perp tangent @ $(9.15, 1.107^c)$.

(2)(b) parallel to initial line $\frac{dy}{d\theta}=0$

$$y = e^{2\theta}\sin\theta$$

$$\frac{dy}{d\theta} = e^{2\theta}\cos\theta + \sin\theta . 2e^{2\theta} = 0$$

$$e^{2\theta}(\cos\theta + 2\sin\theta) = 0$$

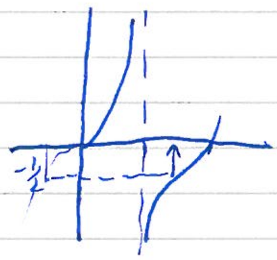
$$\tan\theta = -\frac{1}{2}$$

$$\cancel{\theta = 0.46^c}$$

$$\text{in (1)} \quad \cancel{r = e^{2(0.46)} = 2.53}$$

$\therefore$ ~~parallel tangent @ (2.53, 0.46^c)~~

$$\theta = -0.463, \ \pi - 0.463$$
$$= 2.68$$



$$\text{in (1)} \quad r = e^{2(2.68)} = 211.9$$

$\therefore$ parallel tangent @ $(212, 2.68^c)$.

(3)(a) $r = a\cos 2\theta$

Coords of tangents parallel to initial line when $\frac{dy}{d\theta}=0$

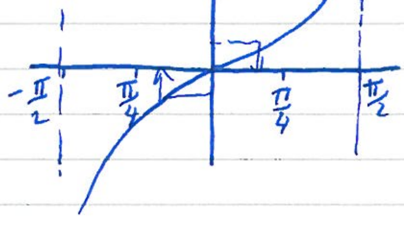
$$y = a\cos 2\theta \sin\theta$$

$$\frac{dy}{d\theta} = a\cos 2\theta . \cos\theta + a\sin\theta . -2\sin 2\theta = 0$$

$$\cos 2\theta \cos\theta = 2\sin\theta \sin 2\theta$$

$$\cot 2\theta = 2\tan\theta$$

$$\frac{1-\tan^2\theta}{2\tan\theta} = 2\tan\theta$$



$$1 - \tan^2\theta = 4\tan^2\theta$$
$$1 = 5\tan^2\theta$$

$$\tan^2\theta = \frac{1}{5}$$

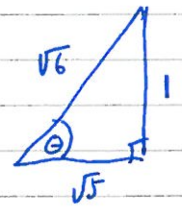
$$\tan\theta = \pm\frac{1}{\sqrt{5}} \qquad \theta = 0.421 \Rightarrow r = \frac{2a}{3}$$
$$\theta = -0.421 \Rightarrow r = \frac{2a}{3}$$

③(a) contd tangents @ $\left(\frac{2a}{3}, \pm 0.421\right)$

(b) Now $y = r\sin\theta$

$y = \frac{2a}{3}\sin(0.421)$ if $\tan\theta = \frac{1}{\sqrt{5}}$

$y = \frac{2a}{3} \times \frac{1}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$



$y = \pm\frac{a\sqrt{6}}{9}$

$\therefore \frac{a\sqrt{6}}{9} = r\sin\theta$

$r = \pm\frac{a\sqrt{6}}{9}\operatorname{Cosec}\theta$

④ $r = a(7 + 2\cos\theta)$ — ①

tangents parallel to initial line when $\frac{dy}{d\theta} = 0$

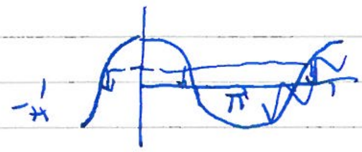
$y = a(7 + 2\cos\theta)\sin\theta$

$y = a(7\sin\theta + 2\sin\theta\cos\theta)$

$y = a(7\sin\theta + \sin 2\theta)$

$\frac{dy}{d\theta} = 7a\cos\theta + 2a\cos 2\theta = 0$

$7a\cos\theta + 4\cos^2\theta - 2 = 0$



$4\cos^2\theta + 7\cos\theta - 2 = 0$

$\cos\theta = -2$ or $\cos\theta = \frac{1}{4}$

$\theta = 1.32, -1.32$

in ① $r = a(7 + 2\cos(1.32)) = \frac{15}{2}a$

$\therefore$ tangents @ $\left(\frac{15a}{2}, \pm 1.32^c\right)$

(5) $r = 2 + \cos\theta$ — (1)

tangents perp to initial line when $\frac{dx}{d\theta} = 0$

$x = 2\cos\theta + \cos^2\theta$

$\frac{dx}{d\theta} = -2\sin\theta - 2\cos\theta\sin\theta = 0$

$2\sin\theta(1 + \cos\theta) = 0$

$\sin\theta = 0 \qquad \cos\theta = -1$

$\theta = 0, \pi \qquad \theta = \pi$

in (1) $\qquad r = 3 \qquad r = 1$

$\therefore$ tangents @ $(3, 0)$ and $(1, \pi)$

Now for equations $x = r\cos\theta$

$x = 3\cos 0 \qquad x = 1\cos\pi$

$x = 3 \qquad x = -1$

$\therefore 3 = r\cos\theta \qquad \therefore -1 = r\cos\theta$

$r = 3\sec\theta \qquad r = -\sec\theta.$

(6) $r = a(1 + \tan\theta)$ — (1)

tangents perp to initial line when $\frac{dx}{d\theta} = 0$

$x = a\left(\cos\theta + \cos\theta . \frac{\sin\theta}{\cos\theta}\right)$

$x = a\cos\theta + a\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta + a\cos\theta = 0$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$ in (1) $r = 2a$ $\therefore$ tangent @ $\left(2a, \frac{\pi}{4}\right)$

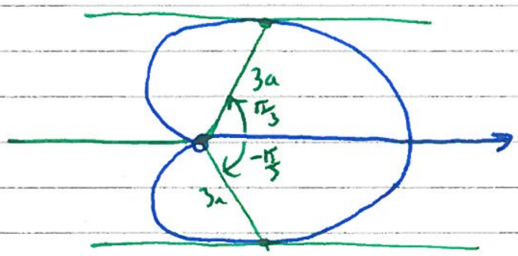
eg 7. (b) contd

$\cos\theta = \frac{1}{2}$ $\qquad$ $\cos\theta = -1$

$\theta = \pm\frac{\pi}{3}$ $\qquad$ $\theta = \pi$

in (1) $\quad r = 2a\left(1 + \frac{1}{2}\right) = 3a$ $\qquad$ $r = 0$

$\therefore$ parallel tangents @ $\left(3a, \pm\frac{\pi}{3}\right)$ and $(0, \pi)$



(c) to find equations ~~tt~~ of tangents parallel use $y = r\sin\theta$, find y & rearrange to give $r = f(\theta)$

$\therefore \quad y = 3a\sin\pm\frac{\pi}{3}$

$y = \pm 3a \cdot \frac{\sqrt{3}}{2}$

$\therefore \quad \pm\frac{3a\sqrt{3}}{2} = r\sin\theta$

$r = \pm\frac{3a\sqrt{3}}{2}\operatorname{cosec}\theta$

or $\quad y = 0$

$\therefore \quad r = 0$