

Higher Derivatives

We are already familiar with the notation for second and third derivatives, given as $\frac{d^2 y}{dx^2}$ and $\frac{d^3 y}{dx^3}$. For a function f given by the equation $y = f(x)$, the 1st, 2nd, 3rd ... n^{th} derivatives are either denoted by

$$\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \frac{d^3 y}{dx^3}, \dots, \frac{d^n y}{dx^n}$$

or by

$$f'(x), f''(x), f'''(x), \dots, f^{(n)}(x)$$

The value of the n^{th} derivative when $x = a$ is given by

$$\left(\frac{d^n y}{dx^n} \right)_a \text{ or } f^{(n)}(a)$$

Eg1 Given that $y = e^{x^3}$, find $\left(\frac{d^3 y}{dx^3} \right)_1$

Exercise 6A Pg 107

Power Series

A series whose terms involve increasing (or decreasing) powers of a variable is called a power series.

We already know that binomial functions of x can be expressed as a series of ascending powers of x , and how the series can be used to find approximate numerical values for certain irrational numbers. There are many other functions of x (such as e^x , $\cos x$, ...) which can be expressed as an infinite series. These series can be used to find approximate values (to any required degree of accuracy) for quantities such as e^2 , $\cos \frac{\pi}{6}$, etc.

MacLaurin's Series

If $f(x)$ is any function of x then, supposing that $f(x)$ can be expanded as a series of ascending powers of x and that this series can be differentiated term by term, we have

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots + a_r x^r + \dots \quad [1]$$

where $a_0, a_1, a_2, \dots$ are constants to be evaluated.

Substituting $x=0$ in (1) $f(0) = a_0$ i.e. $a_0 = f(0)$

Now differentiating (1) w.r.t x

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots \quad [2]$$

and $f'(0) = a_1$

ie $a_1 = f'(0)$

Diff (2) w.r.t x

~~$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \dots$$~~

$$f''(x) = 2a_2 + (3)(2)(a_3x) + (4)(3)(a_4x^2) + (5)(4)(a_5x^3) + \dots \quad [3]$$

hence $f''(0) = 2a_2$

ie $a_2 = \frac{f''(0)}{2}$

Similarly $f'''(x) = (3)(2)a_3 + (4)(3)(2)(a_4)x + (5)(4)(3)(a_5)x^2 + \dots$

So $f'''(0) = 3!a_3$

ie $a_3 = \frac{f'''(0)}{3!}$

~~$$f'''(x) = (4)(3)(2)(a_4) + (5)(4)(3)(2)(a_5)x + \dots$$~~

$\therefore f'''(0) = 4!a_4$

ie $a_4 = \frac{f'''(0)}{4!}$

Hence we can deduce that, after differentiating $f(x)$ r times

$$f^{(r)}(x) = r!a_r + (r+1)(r)\dots(2a_{r+1})x + \dots$$

So $f^{(r)}(0) = r!a_r$

ie $a_r = \frac{f^{(r)}(0)}{r!}$

Substituting these values of $a_0, a_1, a_2, \dots$ in [1] we get

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

This is known as MacLaurin's Series, named after the Scottish mathematician Colin MacLaurin who made extensive use of this series in the 18th century. The series can be found if $f^{(n)}(0)$ exists for all values of n . For the series to be useful, it must converge to $f(x)$. Some functions have a series expansion that is valid for all values of x , but others are only valid for a restricted interval of x . The maths behind the convergence of series is for your delectation in University, for now stating the validity intervals of x is enough.

Eg2 Find the Maclaurin expansion of the following functions:

(a) $f(x) = e^x$ (b) $f(x) = \ln x$ (c) $f(x) = \ln(1+x)$ (d) $f(x) = \cos x$

Eg3 Use Maclaurin's Series to expand the function $f(x) = \tan(e^x)$ as a series of ascending powers of x up to and including x^2 .

eg1

$$y = e^{x^3}$$
$$\frac{dy}{dx} = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

$$\frac{d^2y}{dx^2} = 3x^2 \cdot e^{x^3} \cdot 3x^2 + e^{x^3} \cdot 6x = 9x^4 e^{x^3} + 6x e^{x^3}$$

$$\frac{d^3y}{dx^3} = 9x^4 \cdot e^{x^3} \cdot 3x^2 + e^{x^3} \cdot 36x^3 + 6x \cdot e^{x^3} \cdot 3x^2 + e^{x^3} \cdot 6$$

$$= 27x^6 e^{x^3} + 36x^3 e^{x^3} + 18x^3 e^{x^3} + 6e^{x^3}$$

$$= 27x^6 e^{x^3} + 54x^3 e^{x^3} + 6e^{x^3}$$

$$= e^{x^3} [27x^6 + 54x^3 + 6]$$

$$\text{Now } \left(\frac{d^3y}{dx^3} \right)_1 = e [27 + 54 + 6] = 87e$$

Eg 1 (a) $f(x) = e^x$

Using $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$

$$f(x) = e^x$$

$$f(0) = e^0 = 1$$

$$f'(x) = e^x$$

$$f'(0) = e^0 = 1$$

$$f''(x) = e^x$$

$$f''(0) = e^0 = 1$$

$$\vdots$$

$$f^{(r)}(x) = e^x$$

$$\vdots$$

$$f^{(r)}(0) = e^0 = 1$$

Hence $\frac{f(x)}{e^x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots$

This is valid for all values of x .

(b) $f(x) = \ln x$

$$f(x) = \ln x$$

$$f(0) = \ln 0 \text{ undefined}$$

$\therefore \ln x$ cannot be expanded as a series of ascending powers of x

(c) $f(x) = \ln(1+x)$ $f(0) = \ln 1 = 0$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \quad f'(0) = 1$$

$$f''(x) = -1(1+x)^{-2} \quad f''(0) = -1$$

$$f'''(x) = +2(1+x)^{-3} \quad f'''(0) = 2$$

$$f^{(4)}(x) = (-1)(2)(1+x)^{-4} \quad f^{(4)}(0) = -(3)(2) = -3!$$

$$\vdots$$

$$f^{(r)}(x) = (-1)^{r+1} (r-1)! (1+x)^{-r}$$

$$f^{(r)}(0) = (-1)^{r+1} (r-1)!$$

↑
if r is +ve
↑
-ve

Hence $\ln(1+x) = 0 + x + \frac{(-1)}{2!}x^2 + \frac{(2)}{3!}x^3 + \frac{(-3!)}{4!}x^4 + \dots + \frac{(-1)^{r+1} (r-1)!}{r!}x^r + \dots$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{r+1} x^r}{r} + \dots$$

Valid for $-1 < x \leq 1$. (accept!)

Ex 2 (d) $f(x) = \cos x$ $f(0) = 1$

$f'(x) = -\sin x$ $f'(0) = 0$

$f''(x) = -\cos x$ $f''(0) = -1$

$f'''(x) = +\sin x$ $f'''(0) = 0$

$f^{(4)}(x) = \cos x$ $f^{(4)}(0) = 1$ cycle repeats.

$\therefore \cos x = 1 + (0)x + \frac{(-1)}{2!}x^2 + \frac{(0)}{3!}x^3 + \frac{(1)}{4!}x^4 + \dots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

odd values of r are zero

-ve for odd powers of r

pos for even powers of r

r^{th} term $(-1)^r \frac{x^{2r}}{(2r)!}$

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots$

$$\text{Ex 3} \quad f(x) = \tan e^x \quad f(0) = \tan 1$$

$$f'(x) = \sec^2(e^x) \cdot e^x \quad f'(0) = \sec^2 1$$

$$= e^x [\sec(e^x)]^2$$

$$f''(x) = e^x \cdot 2 [\sec(e^x)] \cdot \sec(e^x) \tan(e^x) \cdot e^x + [\sec(e^x)]^2 \cdot e^x$$

$$f''(0) = 2 \sec 1 \cdot \sec 1 \tan 1 + \sec^2 1$$

$$= 2 \sec^2 1 \tan 1 + \sec^2 1$$

$$= \sec^2 1 [2 \tan 1 + 1]$$

$$\therefore \tan(e^x) = \tan 1 + \underline{\sec^2 1} x + \sec^2 1 [2 \tan 1 + 1] \frac{x^2}{2!}$$

Ex 6A

(1) (a) $f(x) = e^{2x}$

$$f'(x) = e^{2x} \cdot \frac{d}{dx}(2x) = 2e^{2x} = 2f(x)$$

$$f''(x) = 4e^{2x} = 2f'(x) = 2^2 e^{2x}$$

$$f'''(x) = 2f''(x) = 2^3 e^{2x}$$

$$f^n(x) = 2^n e^{2x}$$

(b) $f(x) = (1+x)^n$

$$f'(x) = n(1+x)^{n-1}$$

$$f''(x) = n(n-1)(1+x)^{n-2}$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$$

$$f^n(x) = n(n-1)(n-2)\dots 1(1+x)^{n-n} = n!$$

(c) $f(x) = xe^x$

$$f'(x) = x \cdot e^x + e^x = e^x + xe^x$$

$$f''(x) = e^x + xe^x + e^x = 2e^x + xe^x$$

$$f'''(x) = 2e^x + xe^x + e^x = 3e^x + xe^x$$

$$f^n(x) = ne^x + xe^x$$

(d) $f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x} \cdot 1 = (1+x)^{-1}$$

$$f''(x) = -1(1+x)^{-2} = -(1+x)^{-2}$$

$$f'''(x) = (-1)(-2)(1+x)^{-3} = 2(1+x)^{-3}$$

$$f^n(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$$

$$(2) a) f(x) = y = e^{2+3x}$$

$$f'(x) = 3e^{2+3x}$$

$$f''(x) = 3^2 e^{2+3x}$$

$$f^n(x) = 3^n e^{2+3x}$$

$$(b) f^6(x) = 3^6 e^{2+3x} = 3^6 e^2 e^{3x}$$

$$(f^6(x))_{\ln \frac{1}{9}} = 3^6 e^2 e^{3(\ln \frac{1}{9})} = 3^6 e^2 e^{\ln(\frac{1}{9})^3} = 3^6 e^2 \cdot \frac{1}{9^3} = e^2$$

$$(3) y = \sin^2 3x = (\sin 3x)^2$$

$$(a) \frac{dy}{dx} = 2 \sin 3x \cdot \cos 3x \cdot 3 = 3[2 \sin 3x \cos 3x] = 3 \sin 6x$$

$$(b) \frac{d^2 y}{dx^2} = 3 \cdot \cos 6x \cdot 6 = 18 \cos 6x$$

$$(c) \frac{d^3 y}{dx^3} = -18 \sin 6x \cdot 6 = -108 \sin 6x$$

$$\frac{d^4 y}{dx^4} = -108 \cos 6x \cdot 6 = -648 \cos 6x$$

$$\left(\frac{d^4 y}{dx^4} \right)_{\frac{\pi}{6}} = -648 \cos \pi = 648$$

$$(4) a) f(x) = x^2 e^{-x}$$

$$f'(x) = x^2 \cdot -1e^{-x} + e^{-x} \cdot 2x = (2x - x^2)e^{-x}$$

$$f''(x) = (2x - x^2) \cdot -1e^{-x} + e^{-x}(2 - 2x) = (x^2 - 4x + 2)e^{-x}$$

$$f'''(x) = (x^2 - 4x + 2) \cdot -1e^{-x} + e^{-x}(2x - 4) = (6x - 6 - x^2)e^{-x}$$

As req'd

$$(b) f'''(x) = (6x - 6 - x^2) \cdot -1e^{-x} + e^{-x}(6 - 2x)$$

$$f'''(2) = (12 - 6 - 4) \cdot -1e^{-2} + e^{-2}(6 - 4) = -2e^{-2} + 2e^{-2} = 0$$

5 (a) $y = \sec x$

$$\frac{dy}{dx} = \sec x \tan x$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x \\ &= \sec^3 x + \cancel{\frac{\sin x}{\cos^2 x}} \cdot \tan^2 x \cdot \sec x \end{aligned}$$

$$= \sec^3 x + \sec x [\sec^2 x - 1]$$

$$= 2\sec^3 x - \sec x \quad \text{As required.}$$

(b) $\frac{d^3y}{dx^3} = 6(\sec x)^2 \cdot \sec x \tan x - \sec x \tan x$

$$\left(\frac{d^3y}{dx^3}\right)_{\frac{\pi}{4}} = 6 \left[\sec\left(\frac{\pi}{4}\right)\right]^3 \tan \frac{\pi}{4} - \sec \frac{\pi}{4} \tan \frac{\pi}{4}$$

$$= 6(\sqrt{2})^3 \cdot 1 - \sqrt{2}$$

$$= 12\sqrt{2} - \sqrt{2}$$

$$= 11\sqrt{2} \quad \text{As required.}$$

6 $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$

$$\frac{d^2}{dx^2} \left[2y \frac{dy}{dx} \right] = 2y \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2 \left(\frac{dy}{dx} \right)$$

$$= 2y \frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 \quad \text{As required.}$$

$$\frac{d^3}{dx^3} = \frac{d}{dx} \left[\left(2y \frac{d^2y}{dx^2} \right) + 2 \left(\frac{dy}{dx} \right)^2 \right]$$

$$= \left(2y \cdot \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} \cdot \underset{\text{Product}}{2y} \right) + \left(\underset{\text{Chain rule}}{4 \frac{dy}{dx} \times \frac{d^2y}{dx^2}} \right) = 2y \frac{d^3y}{dx^3} + 2 \left(\frac{dy}{dx} \right) \left(\frac{d^2y}{dx^2} \right) + 4 \left(\frac{dy}{dx} \right) \left(\frac{d^2y}{dx^2} \right)$$

$$= 2 \left[y \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \left(\frac{d^2y}{dx^2} \right) \right] = 2 \left[y \frac{d^3y}{dx^3} + 3 \left(\frac{dy}{dx} \right) \left(\frac{d^2y}{dx^2} \right) \right]$$

⑦ a) $f(x) = \ln[x + \sqrt{1+x^2}]$

$$f'(x) = \frac{1}{x + \sqrt{1+x^2}} \times \left[1 + \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \times 2x \right]$$

$$= \frac{1}{x + \sqrt{1+x^2}} \times \cancel{1+x^2} \left[1 + \frac{x}{\sqrt{1+x^2}} \right]$$

$$= \frac{1}{x + \sqrt{1+x^2}} \times \left[\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}} \right]$$

$$f'(x) = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}}$$

$$\therefore \sqrt{1+x^2} \times f'(x) = \sqrt{1+x^2} \times \frac{1}{\sqrt{1+x^2}} = 1.$$

(b) $f''(x) = -\frac{1}{2}(1+x^2)^{-\frac{3}{2}} \cdot 2x = -x(1+x^2)^{-\frac{3}{2}}$

Now $(1+x^2)f''(x) + xf'(x)$

$$= (1+x^2) \cdot -x(1+x^2)^{-\frac{3}{2}} + x(1+x^2)^{-\frac{1}{2}}$$

$$= -x(1+x^2)^{-\frac{1}{2}} + x(1+x^2)^{-\frac{1}{2}} = 0 \text{ As required}$$

(c) $f'''(x) = -x \cdot -\frac{3}{2}(1+x^2)^{-\frac{5}{2}} \cdot 2x + (1+x^2)^{-\frac{3}{2}} \cdot -1$

$$= 3x^2(1+x^2)^{-\frac{5}{2}} - (1+x^2)^{-\frac{3}{2}}$$

Now $(1+x^2)f'''(x) + 3xf''(x) + f'(x)$

$$= (1+x^2) \left[3x^2(1+x^2)^{-\frac{5}{2}} - (1+x^2)^{-\frac{3}{2}} \right] + 3x \left[-x(1+x^2)^{-\frac{3}{2}} \right] + (1+x^2)^{-\frac{1}{2}}$$

$$= 3x^2(1+x^2)^{-\frac{3}{2}} - (1+x^2)^{-\frac{1}{2}} - 3x^2(1+x^2)^{-\frac{3}{2}} + (1+x^2)^{-\frac{1}{2}}$$

$$= 0 \text{ As req'd.}$$

(d) $f(0) = 1 \quad f'(0) = 0 \quad f'''(0) = -1$

Ex 6B

① (a) $f(x) = (1-x)^{-1}$ $f(0) = 1$

$$f'(x) = -1(1-x)^{-2} \cdot -1 = (1-x)^{-2} \quad f'(0) = 1$$

$$f''(x) = -2(1-x)^{-3} \cdot -1 = 2(1-x)^{-3} \quad f''(0) = 2$$

$$f'''(x) = -6(1-x)^{-4} \cdot -1 = 6(1-x)^{-4} \quad f'''(0) = 6$$

$$\therefore (1-x)^{-1} = 1 + 1x + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \dots$$

$$= 1 + x + x^2 + x^3 + \dots x^n + \dots$$

(b) $f(x) = (1+x)^{\frac{1}{2}}$ $f(0) = 1$

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \cdot 1 \quad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \cdot 1 \quad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-\frac{5}{2}} \quad f'''(0) = \frac{3}{8}$$

$$\therefore \sqrt{1+x} = 1 + \frac{1}{2}x + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^2}{2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!} + \dots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

$$\begin{aligned} \textcircled{2} \quad f(x) &= e^{\sin x} & f(0) &= 1 \\ f'(x) &= \cos x e^{\sin x} & f'(0) &= 1 \\ f''(x) &= \cos x \cdot \cos x e^{\sin x} + e^{\sin x} \cdot (-\sin x) & f''(0) &= 1 - 0 = 1 \end{aligned}$$

$$\therefore e^{\sin x} = 1 + 1x + \frac{1x^2}{2!} = 1 + x + \frac{x^2}{2}$$

$$\begin{aligned} \textcircled{3} \quad (a) \quad f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \end{aligned}$$

$$\begin{aligned} \therefore \cos x &= 1 + 0x + (-1)\frac{x^2}{2!} + 0\frac{x^3}{3!} + (1)\frac{x^4}{4!} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{(-1)^r x^{2r}}{(2r)!} + \dots \quad r=0,1,2,3,\dots \end{aligned}$$

$$(b) \quad 30^\circ = \frac{\pi}{6}$$

$$\therefore \cos x \approx 1 - \frac{\left(\frac{\pi}{6}\right)^2}{2!} + \frac{\left(\frac{\pi}{6}\right)^4}{4!} = 0.86605\dots \text{ which is correct to 3 dp.}$$

$$\textcircled{4} \quad (a) \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

Approximation using n terms to 4 dp are

n	1	2	3	4	5	6	7	8
e	1	2	2.5	2.6667	2.7083	2.7167	2.7181	2.7183

$$\therefore e^1 = 2.718 \text{ to 3 dp.}$$

$$\begin{matrix} 9 \\ 2.7183 \end{matrix}$$

$$(4)(b) \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\ln\left(\frac{6}{5}\right) = \ln\left(1 + \frac{1}{5}\right) \therefore x = 0.2$$

$$\ln\left(\frac{6}{5}\right) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} - \dots$$

n	1	2	3	4	5
$\ln\left(\frac{6}{5}\right)$	0.2	0.18	0.1827	0.1823	0.1823

$$\therefore \ln\left(\frac{6}{5}\right) = 0.182 \text{ to 3 dp.}$$

$$(5)(a) f(x) = e^{3x} \quad f(0) = 1$$

$$f'(x) = 3e^{3x} \quad f'(0) = 3$$

$$f''(x) = (3)(3)e^{3x} \quad f''(0) = 9$$

$$f'''(x) = (3)(3)(3)e^{3x} \quad f'''(0) = 27 \quad f^{(4)}(x) = 3^4 e^{3x} \quad f^{(4)}(0) = 3^4$$

$$\therefore e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{3!} + \frac{3^4 x^4}{4!} + \dots$$

$$= 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8}$$

$$(b) f(x) = \ln(1+2x) \quad f(0) = 0$$

$$f'(x) = \frac{1}{1+2x} \cdot 2 = 2(1+2x)^{-1} \quad f'(0) = 2$$

$$f''(x) = -2(1+2x)^{-2} \cdot 2$$

$$= \frac{-4}{(1+2x)^2} \quad f''(0) = -4$$

$$f'''(x) = -4x - 2(1+2x)^{-3} \cdot 2$$

$$= \frac{16}{(1+2x)^3} \quad f'''(0) = 16$$

$$(5D) \quad f'''(x) = 16x - 3(1+2x)^4 \cdot 2$$

$$= -\frac{96}{(1+2x)^4}$$

$$f'''(0) = -96$$

$$\begin{aligned} \therefore \ln(1+2x) &= 0 + 2x - \frac{4x^2}{2!} + \frac{16x^3}{3!} - \frac{96x^4}{4!} \\ &= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots \end{aligned}$$

$$(c) \quad f(x) = \sin^2 x \quad f(0) = 0$$

$$f'(x) = 2 \sin x \cos x \quad f'(0) = 0$$

$$= \sin 2x$$

$$f''(x) = 2 \cos 2x \quad f''(0) = 2$$

$$f'''(x) = -4 \sin 2x \quad f'''(0) = 0$$

$$f^{(4)}(x) = -8 \cos 2x \quad f^{(4)}(0) = -8$$

$$\therefore \sin^2 x = \frac{2x^2}{2!} - \frac{8x^4}{4!} = x^2 - \frac{x^4}{3}$$

$$(6) \quad \cos\left(x - \frac{\pi}{4}\right) = \cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} [\cos x + \sin x]$$

$$= \frac{1}{\sqrt{2}} \left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + \left(x - \frac{x^3}{3!} + \dots\right) \right]$$

$$= \frac{1}{\sqrt{2}} \left[1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \dots \right] \quad \text{As required}$$

$$(7) a) f(x) = (1-x)^{-\frac{3}{2}} \ln(1-x)$$

$$f'(x) = (1-x)^{-\frac{3}{2}} \cdot \frac{1}{1-x} \cdot -1 + \ln(1-x) \cdot 2(1-x)^{-\frac{3}{2}} \cdot -1$$

$$f'(x) = x-1 - 2(1-x)\ln(1-x)$$

$$f''(x) = 1-0 - \left[2(1-x) \cdot \frac{1}{1-x} \cdot -1 + \ln(1-x) \cdot -2 \right]$$

$$= 1+2+2\ln(1-x)$$

$$= 3+2\ln(1-x) \quad \text{As required}$$

$$(b) f(0) = 0$$

$$f'(0) = -1$$

$$f''(0) = 3$$

$$f'''(x) = 0 + \frac{2}{1-x} \cdot -1 = -\frac{2}{1-x} \quad f'''(0) = -2$$

$$(c) (1-x)^2 \ln(1-x) = -x + \frac{3x^2}{2} - \frac{2x^3}{3!} = -x + \frac{3x^2}{2} - \frac{x^3}{3}$$

$$(8) (a) 3\sin x - 4x \cos x + x$$

$$= 3 \left[x - \frac{x^3}{6} + \frac{x^5}{120} + \dots \right] - 4x \left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots \right] + x$$

$$= 3x - \frac{x^3}{2} + \frac{x^5}{40} - 4x + 2x^3 - \frac{x^5}{6} + x + \dots$$

$$= \frac{3x^3}{2} - \frac{17x^5}{120} + \dots$$

$$(b) \text{ Now as } x \rightarrow 0 \quad \frac{3\sin x - 4x \cos x + x}{x^3} = \frac{\frac{3x^3}{2} - \frac{17x^5}{120} + \dots}{x^3}$$

$$= \frac{3}{2} - \frac{17}{120} x^2 + \dots$$

$$\Rightarrow \frac{3}{2}$$

$$(9) f(x) = \ln(\cos x)$$

$$f(0) = 0$$

$$(a) f'(x) = \frac{1}{\cos x} \cdot -\sin x = -\tan x$$

$$f'(0) = 0$$

$$(b) f''(x) = -\sec^2 x$$

$$f''(0) = -1$$

$$f'''(x) = -2\sec x \cdot \sec x \tan x$$

$$f'''(0) = 0$$

$$= -2\sec^2 x \tan x$$

$$= -2[1 + \tan^2 x] \tan x$$

$$= -2\tan x - 2\tan^3 x$$

$$f^{(4)}(x) = -2\sec^2 x - 6\tan^2 x \cdot \sec^2 x$$

$$f^{(4)}(0) = -2$$

$$(c) \ln \cos x = 0 + 0x + \frac{(-1)x^2}{2!} + 0\frac{x^3}{3!} + \frac{(-2)x^4}{4!} = -\frac{x^2}{2} - \frac{x^4}{12}$$

$$(d) \ln \cos \frac{\pi}{4} = \ln\left(\frac{1}{\sqrt{2}}\right) = \ln 1 - \ln \sqrt{2} = -\frac{1}{2} \ln 2$$

$$\therefore -\left(\frac{\frac{\pi}{4}}{2}\right)^2 - \left(\frac{\frac{\pi}{4}}{12}\right)^4 = -\frac{1}{2} \ln 2$$

$$-\frac{\pi^2}{32} - \frac{\pi^4}{3072} = -\frac{1}{2} \ln 2$$

$$\ln 2 = \frac{\pi^2}{16} + \frac{\pi^4}{1536} = \frac{\pi^2}{16} \left[1 + \frac{\pi^2}{96} \right] \text{ As required.}$$

(10)

$$f(x) = \tan x$$

$$f(0) = 0$$

$$f'(x) = \sec^2 x$$

$$f'(0) = 1$$

$$f''(x) = 2 \sec x \cdot \sec x \tan x$$

$$= 2[1 + \tan^2 x] \tan x$$

$$= 2 \tan x + 2 \tan^3 x$$

$$f''(0) = 0$$

$$f'''(x) = 2 \sec^2 x + 6 \tan^2 x \sec^2 x$$

$$f'''(0) = 2$$

$$= 2 \sec^2 x + 6 [\sec^2 x - 1] \sec^2 x$$

$$= 2 \sec^2 x + 6 \sec^4 x - 6 \sec^2 x$$

$$= 6 \sec^4 x - 4 \sec^2 x$$

$$f^{(4)}(x) = 24 \sec^3 x \cdot \sec x \tan x - 8 \sec x \cdot \sec x \tan x$$

$$f^{(4)}(0) = 0$$

$$= 24 \sec^4 x \tan x - 8 \sec^2 x \tan x$$

$$= 24 \tan x [1 + \tan^2 x]^2 - 8 \tan x [1 + \tan^2 x]$$

$$= 24 \tan x [1 + 2 \tan^2 x + \tan^4 x] - 8 \tan x - 8 \tan^3 x$$

$$= 24 \tan x + 48 \tan^3 x + 24 \tan^5 x - 8 \tan x - 8 \tan^3 x$$

$$= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x$$

$$f^{(5)}(x) = 16 \sec^2 x + 120 \tan^2 x \cdot \sec^2 x + 120 \tan^4 x \sec^2 x$$

$$f^{(5)}(0) = 16$$

$$\therefore \tan x = 0 + (1)x + \frac{(0)x^2}{2!} + \frac{(2)x^3}{3!} + \frac{(0)x^4}{4!} + \frac{16x^5}{5!}$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15}$$

Ans required.

Standard Expansions

The EDEXCEL formulae book contains the Maclaurin's expansions of e^x , $\ln(1+x)$, $\sin x$, $\cos x$ and $\arctan x$. These expansions may be quoted *unless* their derivation is asked for.

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad \text{for } -1 < x \leq 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all } x$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^r \frac{x^{2r+1}}{2r+1} + \dots \quad \text{for } -1 \leq x \leq 1$$

These standard expansions can be used to express some compound functions as series.

Eg4 Expand the following as series of ascending powers of x up to and including the term in x^4

$$(a) \ln \left[\frac{1-2x}{(1+2x)^2} \right] \quad (b) \frac{e^{2x} + e^{-2x}}{e^x}$$

In each case, give the range of values of x for which each expansion is valid.

Eg5 Show that for small values of x , $\sin^2 x \approx x^2 - \frac{1}{3}x^4$

Exercise 6B Pg 114

6C

Eg 4(a) $\ln \left[\frac{1-2x}{(1+2x)^2} \right] = \ln(1-2x) - 2\ln(1+2x)$

Using expansion for $\ln(1+x)$ and replacing x with $(-2x)$...

$$\ln(1-2x) = (-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \dots$$

$$= -2x - 2x^2 - \frac{8x^3}{3} - \frac{4x^4}{1} + \dots - \frac{(-2x)^r}{r} + \dots$$

$$\ln(1+2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots$$

$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots + (-1)^{r+1} \frac{(2x)^r}{r} + \dots$$

$$\text{Now } \ln \left[\frac{1-2x}{(1+2x)^2} \right] = \left[-2x - 2x^2 - \frac{8x^3}{3} - 4x^4 + \dots \right] - 2 \left[2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots \right]$$

$$= -6x + 2x^2 - 8x^3 + 4x^4 + \dots$$

Now for $\ln(1-2x)$ valid for $-1 < (-2x) \leq 1$
 $\frac{1}{2} > x \geq -\frac{1}{2}$

and $\ln(1+2x)$ valid for $-1 < (2x) \leq 1$
 $-\frac{1}{2} < x \leq \frac{1}{2}$

$\therefore$ expansion valid for $-\frac{1}{2} < x < \frac{1}{2}$

Ex 4(b) $\frac{e^{2x} + e^{-2x}}{e^x} = e^x + e^{-3x}$

$$= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right] + \left[1 + (-3x) + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!} + \dots \right]$$

$$= 2 - 2x + \frac{10x^2}{2} - \frac{26x^3}{6} + \frac{82x^4}{24} - \dots$$

$$= 2 - 2x + 5x^2 - \frac{13x^3}{3} + \frac{41x^4}{12} + \dots \quad \text{Valid for all } x.$$

Q5 $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \right]$$

$$= \frac{1}{2} - \frac{1}{2} \left[1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \dots \right]$$

Now if x is small enough to neglect x^6 & higher powers of x then

$$\sin^2 x \approx x^2 - \frac{1}{3}x^4 \text{ as required,}$$

Ex 6C

$$(1) (a) \frac{1}{e^x} = e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots \quad \text{Valid for all } x$$

$$(b) \frac{e^{2x} \times e^{3x}}{e^x} = e^{4x} = 1 + (4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \dots$$

$$= 1 + 4x + 8x^2 + \frac{32x^3}{3} + \dots \quad \text{Valid for all } x$$

$$(c) e^{1+x} = e^1 \times e^x = e \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \right] \quad \text{Valid for all } x$$

$$(d) \ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \dots$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{Valid for } -1 < -x \leq 1$$

$$1 > x \geq -1$$

$$\text{ie } -1 \leq x < 1$$

$$(e) \sin\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right) - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 - \frac{1}{7!}\left(\frac{x}{2}\right)^7 + \dots$$

$$= \frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} - \frac{x^7}{645120} + \dots \quad \text{Valid for all } x$$

$$(f) \ln(2+3x) = \ln 2 \left[1 + \frac{3x}{2} \right] = \ln 2 + \ln\left(1 + \frac{3x}{2}\right)$$

$$= \ln 2 + \left[\left(\frac{3x}{2}\right) - \frac{1}{2}\left(\frac{3x}{2}\right)^2 + \frac{1}{3}\left(\frac{3x}{2}\right)^3 + \dots \right]$$

$$= \ln 2 + \frac{3x}{2} - \frac{9x^2}{8} + \frac{27x^3}{24} + \dots \quad \text{Valid for } -1 < \frac{3x}{2} \leq 1$$

$$\left(+ \frac{9x^3}{8} + \dots \right)$$

$$-\frac{2}{3} < x \leq \frac{2}{3}$$

$$(2) (a) \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\right) - \left(-x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \dots\right)$$

$$= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \dots$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right] \text{ As required}$$

for $\ln(1+x)$ valid for $-1 < x \leq 1$

for $\ln(1-x)$ valid for $-1 < -x \leq 1$

$$1 > x \geq -1 \Rightarrow -1 \leq x < 1$$

to satisfy both $-1 < x < 1$

$$(b) \ln \sqrt{\frac{1+x}{1-x}} = \ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots, -1 < x < 1$$

$$(c) \frac{1+x}{1-x} = \frac{2}{3}$$

$$3+3x = 2-2x$$

$$5x = -1$$

$$x = -\frac{1}{5}$$

$$\therefore \ln\left(\frac{2}{3}\right) = 2\left[-\frac{1}{5} + \frac{1}{3}\left(-\frac{1}{5}\right)^3 + \frac{1}{5}\left(-\frac{1}{5}\right)^5 + \dots\right]$$

$$= -0.4055 \text{ to 4 dp.}$$

$$(d) x = \frac{3}{5} \quad \ln \sqrt{\frac{1+0.6}{1-0.6}} = \ln \sqrt{\frac{1.6}{0.4}} = \ln \sqrt{4} = \ln 2$$

$$\text{Now } \ln 2 = 0.6 + \frac{0.6^3}{3} + \frac{0.6^5}{5} = 0.687552 = 0.69 \text{ to 2 dp}$$

$$\text{a calc, } \ln 2 = 0.693147... = 0.69 \text{ to 2 dp.}$$

$$\begin{aligned} \textcircled{3} \quad e^{2x} - e^{-x} &= \left[1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots \right] - \left[1 + (-x) + \frac{(-x)^2}{2!} + \dots \right] \\ &= 3x + \frac{3x^2}{2} + \text{larger negligible i.e. for small } x \end{aligned}$$

$$\begin{aligned} \textcircled{4(a)} \quad 3x \sin 2x - \cos 3x &= 3x \left[2x - \frac{(2x)^3}{3!} + \dots \right] - \left[1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \dots \right] \\ &= 6x^2 - 4x^4 - 1 + \frac{9x^2}{2} - \frac{27x^4}{8} + \dots \\ &= -1 + \frac{21x^2}{2} - \frac{59x^4}{8} + \dots \end{aligned}$$

$$\begin{aligned} \text{(b) Now } \left(\frac{3x \sin 2x - \cos 3x + 1}{x^2} \right) &= \frac{1}{x^2} \left[-1 + \frac{21x^2}{2} - \frac{59x^4}{8} + 1 + \dots \right] \\ &= \frac{21}{2} - \frac{59}{8}x^2 + \dots \end{aligned}$$

lim as $x \Rightarrow 0$, $\frac{21}{2}$

$$\begin{aligned} \textcircled{5(a)} \quad \ln(1+x-2x^2) &= \ln(1-x)(1+2x) = \ln(1-x) + \ln(1+2x) \\ &= \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) + \left((2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots \right) \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots \\ &= x - \frac{5x^2}{2} + \frac{7x^3}{3} - \frac{17x^4}{4} + \dots \end{aligned}$$

Q2. (a) $\ln(1+x)$ valid for $-1 < x \leq 1$

$\ln(1+2x)$ valid for $-\frac{1}{2} < x \leq \frac{1}{2}$ $\therefore$ whole valid for $-\frac{1}{2} < x \leq \frac{1}{2}$

$$\textcircled{5} \text{ (b)} \ln(9+6x+x^2) = \ln(x+3)^2 = 2\ln(3+x) = 2\ln 3 \left(1+\frac{x}{3}\right) \\ = 2\ln 3 + 2\ln\left(1+\frac{x}{3}\right)$$

$$= 2\ln 3 + 2 \left[\left(\frac{x}{3}\right) - \frac{1}{2}\left(\frac{x}{3}\right)^2 + \frac{1}{3}\left(\frac{x}{3}\right)^3 - \frac{1}{4}\left(\frac{x}{3}\right)^4 + \dots \right] \\ = 2\ln 3 + \frac{2x}{3} - \frac{x^2}{9} + \frac{2x^3}{81} - \frac{1}{162}x^4 + \dots$$

$$\text{Valid for } -1 < \frac{x}{3} \leq 1 \Rightarrow -3 < x \leq 3$$

$$\textcircled{6} \text{ (a)} \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} + \dots \\ = 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} + \dots$$

$$\text{(b)} \cos 2x = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1}{2} [1 - \cos 2x]$$

$$= \frac{1}{2} \left[1 - \left(1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} + \dots \right) \right]$$

$$= x^2 - \frac{1}{3}x^4 + \frac{2x^6}{45} - \frac{1}{315}x^8 + \dots$$

$$\textcircled{7} \ln(1+x) + (x-1)(e^x-1)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) + (x-1) \left(-1 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right)$$

$$= \left(\cancel{x} - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) + \left(\cancel{x^2} + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} - \cancel{x} - \cancel{\frac{x^2}{2}} - \frac{x^3}{6} - \frac{x^4}{24} + \dots \right)$$

$$= \frac{2x^3}{3} - \frac{1}{8}x^4$$

$$\text{o.o } p = \frac{2}{3} \quad q = -\frac{1}{8}$$

$$(8) a) \frac{\sin x}{(1-x)^2} = \sin x (1-x)^{-2}$$

$$\sin x = x - \frac{x^3}{6} + \dots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

↓

$$(C4) \quad (1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)(-x)^2}{2!} + \frac{(-2)(-3)(-4)(-x)^3}{3!} + \dots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$\therefore \frac{\sin x}{(1-x)^2} = \left(x - \frac{x^3}{6} + \dots \right) (1 + 2x + 3x^2 + 4x^3 + \dots)$$

$$= x + 2x^2 + 3x^3 + 4x^4 - \frac{x^3}{6} - \frac{x^4}{3} + \dots$$

$$= x + 2x^2 + \frac{17x^3}{6} + \frac{11x^4}{3} + \dots$$

$$(b) \quad y = x + 2x^2 + \frac{17x^3}{6} + \frac{11x^4}{3} + \dots$$

$$\frac{dy}{dx} = 1 + 4x + \dots$$

$$@ x=0 \quad \frac{dy}{dx} = 1.$$

$$(9) a) (1-3x) \ln(1+2x) = (1-3x) \left(2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots \right)$$

$$= (1-3x) \left(2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \dots \right)$$

$$= 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 - 6x^2 + 6x^3 - 8x^4 + \dots$$

$$= 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4 + \dots$$

$$\begin{aligned}
 (9) \text{ b) } e^{2x} \sin x &= \left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots \right) \left(x - \frac{x^3}{3!} + \dots \right) \\
 &= x + 2x^2 + 2x^3 + \frac{4x^4}{3} - \frac{x^3}{6} - \frac{x^4}{3} + \dots \\
 &= x + 2x^2 + \frac{11}{6}x^3 + x^4 + \dots
 \end{aligned}$$

$$\begin{aligned}
 (c) \sqrt{1+x^2} e^{-x} &= (1+x^2)^{\frac{1}{2}} e^{-x} \\
 &\stackrel{\text{CH}}{=} \left[1 + \left(\frac{1}{2}\right)(x^2) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(x^2)^2}{2!} + \dots \right] \left[1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \dots \right] \\
 &= \left(1 + \frac{x^2}{2} - \frac{x^4}{8} \right) \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} \right) \\
 &= 1 + \frac{x^2}{2} - \frac{x^4}{8} - x - \frac{x^3}{2} + \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^3}{6} + \frac{x^4}{24} \\
 &= 1 - x + x^2 - \frac{2x^3}{3} + \frac{1}{6}x^4 + \dots
 \end{aligned}$$

$$\begin{aligned}
 (10) \quad e^{-\frac{x^2}{2}} &= 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{2}\right)^3 + \frac{1}{4!}\left(-\frac{x^2}{2}\right)^4 + \dots \\
 &= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} + \dots
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \int_{-1}^1 e^{-\frac{x^2}{2}} dx &= \int_{-1}^1 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} \right) dx \\
 &= \left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} \right]_{-1}^1 \\
 &= 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} - \left(-1 + \frac{1}{6} - \frac{1}{40} + \frac{1}{336} - \frac{1}{3456} \right) \\
 &= 2 - \frac{2}{6} + \frac{2}{40} - \frac{2}{336} + \frac{2}{3456} \\
 &= 1.711 \text{ to 3 dp}
 \end{aligned}$$

$$(11) a) e^{px} \sin 3x = \left(1 + px + \frac{(px)^2}{2!} + \dots\right) \left(3x - \frac{(3x)^3}{3!} + \dots\right)$$

$$= \left(1 + px + \frac{p^2 x^2}{2} + \dots\right) \left(3x - \frac{9x^3}{2} + \dots\right)$$

$$= 3x + 3px^2 + 3\frac{p^2 x^3}{2} - \frac{9x^3}{2} + \dots$$

$$= 3x + 3px^2 + 3\left(\frac{p^2 - 3}{2}\right)x^3 + \dots$$

$$b) e^{px} \sin 3x + \ln(1+qx) - x$$

$$\left(3x + 3px^2 + 3\left(\frac{p^2 - 3}{2}\right)x^3 + \dots\right) + \left(qx - \frac{q^2 x^2}{2} + \frac{q^3 x^3}{3} + \dots\right) - x$$

$$(2+q)x + \left(3p - \frac{q^2}{2}\right)x^2 + \left(3\left(\frac{p^2 - 3}{2}\right) + \frac{q^3}{3}\right)x^3$$

Now x term is zero $\therefore 2+q=0 \quad q=-2$

x^2 term is zero. $3p - \frac{(-2)^2}{2} = 0$

$$3p - 2 = 0$$

$$p = \frac{2}{3}$$

and x^3 term = $k \quad \frac{3}{2} \left(\left(\frac{2}{3}\right)^2 - 3\right) + \frac{(-2)^3}{3} = k$

$$\frac{3}{2} \left(\frac{4}{9} - 3\right) - \frac{8}{3} = k$$

$$k = -\frac{13}{2}$$

$$(12)(a) f(x) = e^{x - \ln x} \sin x = e^x e^{-\ln x} \sin x = \frac{e^x \sin x}{x}$$

$$f(x) = \frac{1}{x} \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \right) \left(x - \frac{x^3}{6} + \dots \right)$$

$$= \frac{1}{x} \left(x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^3}{6} - \frac{x^4}{6} + \dots \right)$$

$$= \frac{1}{x} \left(x + x^2 + \frac{x^3}{3} + \dots \right)$$

$$= 1 + x + \frac{x^2}{3} \quad \Delta \text{ required}$$

$$(b) f(0.1) = \frac{e^{0.1} \sin 0.1}{0.1} = 1.103329\dots \approx 1.10333 \text{ to 6 sf.}$$

$$f(0.1) \approx 1 + 0.1 + \frac{(0.1)^2}{3} = 1.10333\dots \approx 1.10333 \text{ to 6 sf.}$$

Taylor's Expansion

Maclaurin's expansion cannot be applied to all problems. For instance we have shown that $\ln x$ cannot be expanded as a series of ascending powers of x because $\ln x$ is undefined when $x = 0$.

Also the Maclaurin expansion of $\ln(x+1)$ is valid only for $-1 < x \leq 1$ and the series so produced cannot be used to find, say, $\ln(3)$.

Problems such as these can sometimes be overcome by expanding $f(x)$ as a series of ascending powers of $(x-a)$, ie

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots$$

Following a similar process to that which we used to obtain Maclaurin's expansion, we get:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(r)}(a)}{r!}(x-a)^r + \dots$$

This is known as Taylor's Expansion. By replacing x with $x+a$, it can also be written:

$$f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \dots + \frac{f^{(r)}(a)}{r!}x^r + \dots$$

At this level, we can assume that any Taylor Series is convergent for values of x close to a .

Eg6 Expand $\ln x$ as a series of $(x-a)$ up to the term containing $(x-a)^3$. Use your expression to evaluate $\ln 1.01$ correct to five decimal places.

Eg7 Using Taylor's series, expand $x^2 \ln x$ in ascending powers of $(x-1)$ up to and including the term in $(x-1)^3$.

Eg8 Using Taylor's series, expand $\sin\left(x + \frac{\pi}{6}\right)$ in ascending powers of x up to and including the term in x^3 .

Exercise 6D Pg 117

Eg6 $f(x) = \ln x$ $f(a) = \ln a$

$f'(x) = \frac{1}{x} = x^{-1}$ $f'(a) = \frac{1}{a}$

$f''(x) = -\frac{1}{x^2}$ $f''(a) = -\frac{1}{a^2}$

$f'''(x) = +\frac{2}{x^3}$ $f'''(a) = \frac{2}{a^3}$

$\therefore \ln x = \ln a + \frac{1}{a}(x-a) + \left(-\frac{1}{a^2}\right)\frac{(x-a)^2}{2!} + \left(\frac{2}{a^3}\right)\frac{(x-a)^3}{3!} + \dots$

Now if $x = 1.01$ and we let $a = 1$, then $x - a = 0.01$ which is small

Hence $\ln x = \ln 1 + \frac{1}{1}(0.01) - \frac{1}{1^2}\frac{(0.01)^2}{2} + \frac{2}{1^3}\frac{(0.01)^3}{6} + \dots$

$= 0 + 0.01 - 0.00005 + \dots$

$= 0.00995$ correct to 5 dp.

Eg7 ~~$f(x) = x^2 \ln x$~~ ~~$a = 1$~~

Eg7 Using Taylor Series with $f(x) = x^2 \ln x$ and $a = 1$

$f(x) = x^2 \ln x$ $f(1) = 0$

$f'(x) = x^2 \times \frac{1}{x} + 2x \ln x$
 $= x + 2x \ln x$ $f'(1) = 1$

$f''(x) = 1 + 2x \cdot \frac{1}{x} + 2 \ln x$
 $= 3 + 2 \ln x$ $f''(1) = 3$

$f'''(x) = \frac{2}{x}$ $f'''(1) = 2$

$\therefore x^2 \ln x = 0 + 1(x-1) + \frac{3}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \dots$
 $= (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots$

Eg 8 $f(x) = \sin x$ Then $f(x + \frac{\pi}{6}) = \sin(x + \frac{\pi}{6})$

∴ Use Verin (2) of Taylor expansion with $f(x) = \sin x$ $a = \frac{\pi}{6}$

$$f(x) = \sin x \quad f(a) = \sin \frac{\pi}{6} = \frac{1}{2}$$

$$f'(x) = \cos x \quad f'(a) = \frac{\sqrt{3}}{2}$$

$$f''(x) = -\sin x \quad f''(a) = -\frac{1}{2}$$

$$f'''(x) = -\cos x \quad f'''(a) = -\frac{\sqrt{3}}{2}$$

$$\therefore \sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}x + \left(-\frac{1}{2}\right)\frac{x^2}{2!} + \left(-\frac{\sqrt{3}}{2}\right)\frac{x^3}{3!} + \dots$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}x - \frac{x^2}{4} - \frac{\sqrt{3}}{12}x^3 + \dots$$

Ex 6)

① a) $f(x) = x^{\frac{1}{2}}$ $a = 1$

$$f(x) = x^{\frac{1}{2}} = f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^{(4)}(1)(x-1)^4}{4!} + \dots$$

$$f(1) = 1$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \quad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \quad f''(1) = -\frac{1}{4}$$

$$f'''(x) = +\frac{3}{8}x^{-\frac{5}{2}} \quad f'''(1) = \frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16}x^{-\frac{7}{2}} \quad f^{(4)}(1) = -\frac{15}{16}$$

$$\therefore \sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \dots$$

$$(b) \sqrt{1.2} \approx 1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 + \frac{1}{16}(0.2)^3 - \frac{5}{128}(0.2)^4 = 1.095 \text{ (3dp)}$$

② a) $f(x) = \ln x$ $a = e$

$$f(x) = \ln x = f(e) + f'(e)(x-e) + \frac{f''(e)(x-e)^2}{2!} + \dots$$

$$f(x) = \ln x$$

$$f(e) = 1$$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f'(e) = e^{-1}$$

$$f''(x) = -x^{-2}$$

$$f''(e) = -e^{-2}$$

$$\therefore \ln x \approx 1 + \frac{x-e}{e} - \frac{(x-e)^2}{2e^2} + \dots$$

②/b) $f(x) = \tan x$ $a = \frac{\pi}{3}$

$$f(x) = \tan x = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)(x - \frac{\pi}{3}) + \frac{f''\left(\frac{\pi}{3}\right)(x - \frac{\pi}{3})^2}{2!} + \frac{f'''\left(\frac{\pi}{3}\right)(x - \frac{\pi}{3})^3}{3!} + \dots$$

$$f(x) = \tan x \quad f\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$f'(x) = \sec^2 x \quad f'\left(\frac{\pi}{3}\right) = 4$$

$$f''(x) = 2 \sec x \sec x \tan x$$

$$= 2 \sec^2 x \tan x \quad f''\left(\frac{\pi}{3}\right) = 8\sqrt{3}$$

$$f'''(x) = 2 \sec^2 x \cdot \sec^2 x + \tan x \cdot 4 \sec x \cdot \sec x \tan x = 32 + 48 = 80$$

$$\begin{aligned} \therefore f(x) = \tan x &= \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + \frac{8\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)^2 + \frac{80}{6}\left(x - \frac{\pi}{3}\right)^3 + \dots \\ &= \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + 4\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{3}\right)^3 + \dots \end{aligned}$$

(c) $f(x) = \cos x$ $a = 1$

$$\cos x = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^{(4)}(1)(x-1)^4}{4!} + \dots$$

$$f(x) = \cos x \quad f(1) = \cos 1$$

$$f'(x) = -\sin x \quad f'(1) = -\sin 1$$

$$f''(x) = -\cos x \quad f''(1) = -\cos 1$$

$$f'''(x) = \sin x \quad f'''(1) = \sin 1$$

$$f^{(4)}(x) = \cos x \quad f^{(4)}(1) = \cos 1$$

$$\therefore \cos x \approx \cos 1 - (\sin 1)(x-1) - \frac{(\cos 1)(x-1)^2}{2} + \frac{(\sin 1)(x-1)^3}{6} + \frac{(\cos 1)(x-1)^4}{24} + \dots$$

$$(3)(a)(i) \quad f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f'''(a)}{3!}x^3 + \dots$$

$$f(x) = \cos x \quad \text{Hence} \quad f\left(x + \frac{\pi}{4}\right) = \cos\left(x + \frac{\pi}{4}\right) \quad a = \frac{\pi}{4}$$

$$f(x) = \cos x \quad f(a) = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin x \quad f'(a) = -\frac{\sqrt{2}}{2}$$

$$f''(x) = -\cos x \quad f''(a) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = \sin x \quad f'''(a) = \frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \cos x \quad f^{(4)}(a) = \frac{\sqrt{2}}{2}$$

$$\therefore \cos\left(x + \frac{\pi}{4}\right) \approx \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2} \times \frac{1}{2}x^2 + \frac{\sqrt{2}}{2} \times \frac{1}{6}x^3 + \frac{\sqrt{2}}{2} \times \frac{1}{24}x^4 + \dots$$

$$\approx \frac{\sqrt{2}}{2} \left[1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots \right]$$

$$(ii) \quad f(x) = \ln x \quad f(x+5) = \ln(x+5) \quad a=5$$

$$f(x) = \ln x \quad f(5) = \ln 5$$

$$f'(x) = \frac{1}{x} = x^{-1} \quad f'(5) = \frac{1}{5}$$

$$f''(x) = -x^{-2} \quad f''(5) = -\frac{1}{25}$$

$$f'''(x) = 2x^{-3} \quad f'''(5) = \frac{2}{125}$$

$$f^{(4)}(x) = -6x^{-4} \quad f^{(4)}(5) = -\frac{6}{625}$$

$$\therefore \ln(x+5) \approx \ln 5 + \frac{1}{5}x - \frac{1}{25} \times \frac{1}{2}x^2 + \frac{2}{125} \times \frac{1}{6}x^3 - \frac{6}{625} \times \frac{1}{24}x^4 + \dots$$

$$= \ln 5 + \frac{1}{5}x - \frac{1}{50}x^2 + \frac{1}{375}x^3 - \frac{1}{2500}x^4 + \dots$$

$$(3)(a)(iii) \quad f(x) = \sin x \quad f(x + \frac{\pi}{3}) = \sin(x - \frac{\pi}{3}) \quad a = -\frac{\pi}{3}$$

$$f(x) = \sin x \quad f(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$$

$$f'(x) = \cos x \quad f'(-\frac{\pi}{3}) = \frac{1}{2}$$

$$f''(x) = -\sin x \quad f''(-\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

$$f'''(x) = -\cos x \quad f'''(-\frac{\pi}{3}) = -\frac{1}{2}$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$$

$$\begin{aligned} \therefore \sin(x - \frac{\pi}{3}) &\approx -\frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3}) + \frac{\sqrt{3}}{2} \times \frac{1}{2}(x - \frac{\pi}{3})^2 - \frac{1}{2} \times \frac{1}{6}(x - \frac{\pi}{3})^3 + \frac{\sqrt{3}}{2} \times \frac{1}{24}(x - \frac{\pi}{3})^4 + \dots \\ &\approx \frac{1}{2} \left[-\sqrt{3} + (x - \frac{\pi}{3}) + \frac{\sqrt{3}}{2}(x - \frac{\pi}{3})^2 - \frac{1}{6}(x - \frac{\pi}{3})^3 + \frac{\sqrt{3}}{24}(x - \frac{\pi}{3})^4 + \dots \right] \end{aligned}$$

$$\sin(x - \frac{\pi}{3}) \approx -\frac{\sqrt{3}}{2} + \frac{1}{2}x + \frac{\sqrt{3}}{2} \times \frac{1}{2}x^2 - \frac{1}{2} \times \frac{1}{6}x^3 - \frac{\sqrt{3}}{2} \times \frac{1}{24}x^4 + \dots$$

$$\approx \frac{1}{2} \left[-\sqrt{3} + x + \frac{\sqrt{3}}{2}x^2 - \frac{x^3}{6} - \frac{\sqrt{3}}{24}x^4 + \dots \right]$$

(b) For ln 5.2 $x = 0.2$

$$\approx \ln 5 + \frac{1}{5} \times 0.2 - \frac{1}{50}(0.2)^2 + \frac{1}{375}(0.2)^3 - \frac{1}{2500}(0.2)^4$$

$$\approx 1.649 \text{ (4.s.f.)}$$

④ a) $y = xe^x$

$$\frac{dy}{dx} = x \cdot e^x + e^x \cdot 1 = (1+x)e^x$$

$$\frac{d^2y}{dx^2} = (1+x)e^x + e^x \cdot 1 = (2+x)e^x$$

$$\frac{d^3y}{dx^3} = (2+x)e^x + e^x \cdot 1 = (3+x)e^x$$

$$\therefore \frac{d^ny}{dx^n} = (n+x)e^x \quad \text{each extra diff increases result by } e^x$$

(b) $f(x) = xe^x \quad a = -1 \quad f(-1) = -e^{-1}$

$$f'(x) = (1+x)e^x \quad f'(-1) = 0$$

$$f''(x) = (2+x)e^x \quad f''(-1) = e^{-1}$$

$$f'''(x) = (3+x)e^x \quad f'''(-1) = 2e^{-1}$$

$$f^{(4)}(x) = (4+x)e^x \quad f^{(4)}(-1) = 3e^{-1}$$

$$\text{Now } xe^x = -e^{-1} + 0(x+1) + \frac{e^{-1}}{2!}(x+1)^2 + \frac{2e^{-1}}{3!}(x+1)^3 + \frac{3e^{-1}}{4!}(x+1)^4 + \dots$$

$$= e^{-1} \left[-1 + \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 + \frac{1}{8}(x+1)^4 + \dots \right]$$

⑤ $f(x) = x^3 \ln x \quad a = 1 \quad f(a) = 0$

$$f'(x) = \cancel{x^3} \times \frac{1}{x} + 3x^2 \ln x = x^2$$

$$f'(x) = x^3 \times \frac{1}{x} + 3x^2 \ln x = x^2(1 + 3 \ln x) \quad f'(1) = 1$$

$$f''(x) = x^2 \times \frac{3}{x} + \cancel{2x} (1 + 3 \ln x) = 3x + 2x + 6x \ln x = x(5 + 6 \ln x) \quad f''(1) = 5$$

$$f'''(x) = x \times \frac{6}{x} + (5 + 6 \ln x) = 11 + 6 \ln x \quad f'''(1) = 11$$

$$f^{(4)}(x) = \frac{6}{x} \quad f^{(4)}(1) = 6$$

$$\begin{aligned}
 (5) \quad x^3 \ln x &= 0 + 1(x-1) + 5 \times \frac{1}{2}(x-1)^2 + 11 \times \frac{1}{6}(x-1)^3 + 6 \times \frac{1}{24}(x-1)^4 + \dots \\
 &= (x-1) + \frac{5}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + \frac{1}{4}(x-1)^4 + \dots
 \end{aligned}$$

$$(b) \quad (1.5)^3 \ln(1.5) = 0.5 + \frac{5}{2}(0.5)^2 + \frac{11}{6}(0.5)^3 + \frac{1}{4}(0.5)^4 + \dots$$

$$(1.5)^3 \ln(1.5) = 1.3697 \dots$$

$$\ln(1.5) = \frac{1.3697 \dots}{(1.5)^3} = 0.4059 \text{ to 4dp.}$$

$$(6) \quad f(x) = \tan x \quad f(x-\alpha) = \tan(x-\alpha)$$

$$f(x) = \tan x \quad f(-\alpha) = \tan\left(-\arctan\left(\frac{3}{4}\right)\right) = -\frac{3}{4}$$

$$f'(x) = \sec^2 x \quad f'(-\alpha) = \sec^2\left(-\arctan\left(\frac{3}{4}\right)\right) = \frac{25}{16}$$

$$f''(x) = 2 \sec x \cdot \sec x \tan x$$

$$= 2 \sec^2 x \tan x = 2 \times \frac{25}{16} \times -\frac{3}{4} = -\frac{75}{32}$$

$$\text{Now } \tan\left(x - \arctan\left(\frac{3}{4}\right)\right) = -\frac{3}{4} + \frac{25}{16}x - \frac{75}{32} \times \frac{1}{2}x^2 + \dots$$

$$= -\frac{3}{4} + \frac{25}{16}x - \frac{75}{64}x^2 + \dots$$

$$(7) \quad f(x) = \sin 2x \quad a = \frac{\pi}{6} \quad f\left(\frac{\pi}{6}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$f'(x) = 2 \cos 2x \quad f'(a) = +1$$

$$f''(x) = -4 \sin 2x \quad f''(a) = -2\sqrt{3}$$

$$f'''(x) = -8 \cos 2x \quad f'''(a) = -4\sqrt{3}$$

$$f^{(4)}(x) = 16 \sin 2x \quad f^{(4)}(a) = 8\sqrt{3}$$

$$\begin{aligned} \sin 2x &= \frac{\sqrt{3}}{2} - 1x + \frac{-2\sqrt{3}}{2}x^2 + -4 \times \frac{1}{6}x^3 + 8\sqrt{3} \times \frac{1}{24}x^4 + \dots \\ &= \frac{\sqrt{3}}{2} - x - \sqrt{3}x^2 - \frac{2}{3}x^3 + \frac{\sqrt{3}}{3}x^4 + \dots \end{aligned}$$

$$\sin 2x = \frac{\sqrt{3}}{2} - 1\left(x - \frac{\pi}{6}\right) - \frac{2\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)^2 - 4 \times \frac{1}{6}\left(x - \frac{\pi}{6}\right)^3 + 8\sqrt{3} \times \frac{1}{24}\left(x - \frac{\pi}{6}\right)^4 + \dots$$

$$= \frac{\sqrt{3}}{2} + \left(x - \frac{\pi}{6}\right) - \sqrt{3}\left(x - \frac{\pi}{6}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)^4 + \dots$$

$$(8) \quad y = (1+x)^{-1/2}$$

$$(a) \quad \frac{dy}{dx} = -\frac{1}{2}(1+x)^{-3/2} \quad \left(\frac{dy}{dx}\right)_3 = -\frac{1}{2} \times \frac{1}{(\sqrt{4})^3} = -\frac{1}{16}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{1}{2} \times -\frac{3}{2}(1+x)^{-5/2} = \frac{3}{4}(1+x)^{-5/2} \quad \left(\frac{d^2y}{dx^2}\right)_3 = \frac{3}{4} \times \frac{1}{(\sqrt{4})^5} = \frac{3}{4} \times \frac{1}{32} \\ &= \frac{3}{128} \end{aligned}$$

$$(b) \quad f(x) = (1+x)^{-1/2} \quad a = 3 \quad f(3) = \frac{1}{2}$$

$$\frac{1}{\sqrt{1+x}} = +\frac{1}{2} + \frac{-1}{16}(x-3) + \frac{3}{128} \times \frac{1}{2}(x-3)^2 + \dots$$

$$= +\frac{1}{2} - \frac{1}{16}(x-3) + \frac{3}{256}(x-3)^2 + \dots$$

Using Taylor's Series to approximate the solution of Differential Equations

We have already looked at some analytical methods for solving a few specific types of differential equation. In practical situations where problems involving rates of change give rise differential equations however, these are rarely of a type that can be solved analytically. So the need arises for numerical methods which provide approximate solutions.

The use of numerical methods to solve differential equations is a vast topic, and at our level, we need only give consideration to one such method – the use of Taylor's Series.

Consider the Taylor's expansion of $f(x)$ about $x = x_0$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$

By truncating the series after a certain number of terms we get a polynomial function that approximates to $f(x)$. How good this approximation is depends on a number of factors such as, the number of terms included, how close x is to x_0 and so on.

It is not necessary to know $f(x)$ in order to derive the series. Provided initial conditions are known, the series can be derived from a differential equation.

Differential equations are usually formulated in terms of $y, x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$, where $y = f(x)$. This would make the above Taylor expansion look like:

$$y = y_0 + \left(\frac{dy}{dx}\right)_{x_0} (x - x_0) + \left(\frac{d^2y}{dx^2}\right)_{x_0} \frac{(x - x_0)^2}{2!} + \left(\frac{d^3y}{dx^3}\right)_{x_0} \frac{(x - x_0)^3}{3!} + \dots$$

Where $y_0, \left(\frac{dy}{dx}\right)_{x_0}, \left(\frac{d^2y}{dx^2}\right)_{x_0}, \dots$ are the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$ when $x = x_0$

Eg9 Find, as a series of ascending powers of x up to and including the term in x^4 , an approximate solution to the differential equation

$$\frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + y^2 = 0$$

Where $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$

Exercise 6E Pg 122

eqn $\frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + y^2 = 0 \quad \text{--- (1)}$

given that $x_0 = 0, y_0 = 1 \quad \left(\frac{dy}{dx}\right)_0 = 2$

with $x_0 = 0$ the Taylor Series up to the term in x^4 becomes:

$$y \approx y_0 + \left(\frac{dy}{dx}\right)_0 x + \left(\frac{d^2 y}{dx^2}\right)_0 \frac{x^2}{2!} + \left(\frac{d^3 y}{dx^3}\right)_0 \frac{x^3}{3!} + \left(\frac{d^4 y}{dx^4}\right)_0 \frac{x^4}{4!} \quad \text{--- (2)}$$

Sub in (1) $\left(\frac{d^2 y}{dx^2}\right)_0 + 2(0)(2) + 1^2 = 0$
 $\left(\frac{d^2 y}{dx^2}\right)_0 = -1$

differentiating (1) $\frac{d^3 y}{dx^3} + \left[2x \cdot \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx}\right] + \left[2y \frac{dy}{dx}\right] = 0 \quad \text{--- (3)}$

So $\left(\frac{d^3 y}{dx^3}\right)_0 + 2(0)(-1) + 2(2) + 2(1)(2) = 0$
 $\left(\frac{d^3 y}{dx^3}\right)_0 = -8$

differentiating (3)

$$\frac{d^4 y}{dx^4} + \left[2x \cdot \frac{d^3 y}{dx^3} + 2 \frac{d^2 y}{dx^2}\right] + 2 \frac{dy}{dx} + \left[2y \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 2 \frac{dy}{dx}\right] = 0$$

$$\frac{d^4 y}{dx^4} + 4 \frac{d^2 y}{dx^2} + 2x \frac{d^3 y}{dx^3} + 2y \frac{d^2 y}{dx^2} + 2 \left(\frac{dy}{dx}\right)^2 = 0$$

So $\left(\frac{d^4 y}{dx^4}\right)_0 + 4(-1) + 2(0)(-8) + 2(1)(-1) + 2(2)^2 = 0$

$$\left(\frac{d^4 y}{dx^4}\right)_0 = 4 + 0 + 2 - 8 = -2 \quad (-4 \text{ in eq in book})$$

Now in (2)

$$y \approx 1 + 2x + \frac{(-1)x^2}{2} + \frac{(-8)x^3}{6} + \frac{(-2)x^4}{24}$$

$$y \approx 1 + 2x - \frac{x^2}{2} - \frac{4x^3}{3} - \frac{1}{12}x^4 \quad \left(-\frac{1}{6} \text{ in eq in book}\right)$$

Ex 6E

$$\textcircled{1} \quad \frac{d^2 y}{dx^2} = x + 2y$$

$$\text{given that } x_0 = 0 \quad y_0 = 1 \quad \left(\frac{dy}{dx}\right)_0 = \frac{1}{2}$$

$$\left(\frac{d^2 y}{dx^2}\right)_0 = 0 + 2 = 2$$

$$\frac{d^3 y}{dx^3} = 1 + 2 \frac{dy}{dx}$$

$$\left(\frac{d^3 y}{dx^3}\right)_0 = 1 + 2\left(\frac{1}{2}\right) = 2$$

$$\frac{d^4 y}{dx^4} = 2 \frac{d^2 y}{dx^2}$$

$$\left(\frac{d^4 y}{dx^4}\right)_0 = 4$$

$$\text{Now } y = y_0 + x \left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!} \left(\frac{d^2 y}{dx^2}\right)_0 + \frac{x^3}{3!} \left(\frac{d^3 y}{dx^3}\right)_0 + \frac{x^4}{4!} \left(\frac{d^4 y}{dx^4}\right)_0 + \dots$$

$$y = 1 + \frac{x}{2} + x^2 + \frac{x^3}{6} \times 2 + \frac{x^4}{24} \times 4 + \dots$$

$$y = 1 + \frac{x}{2} + x^2 + \frac{x^3}{3} + \frac{x^4}{6} + \dots$$

$$(2) \quad (1+x^2) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 0 \quad - (1)$$

given that $x_0=0$ $y_0=0$ $\left(\frac{dy}{dx}\right)_0 = 1$

$$u(1) \quad \left(\frac{d^2 y}{dx^2}\right)_0 = 0$$

$$\text{diff } \left((1+x^2) \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 2x \right) + \left(x \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \right) = 0$$

$$(1+x^2) \frac{d^3 y}{dx^3} + 3x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$$

$$\text{Sub } \left(\frac{d^3 y}{dx^3}\right)_0 + 0 + 1 = 0 \Rightarrow \left(\frac{d^3 y}{dx^3}\right)_0 = -1$$

$$\text{Now } y = 0 + x(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(-1)$$

$$y = x - \frac{x^3}{6} + \dots$$

$$(3) \quad \frac{dy}{dx} + y - e^x = 0 \quad - (1)$$

given that $x_0=0$ $y_0=2$

$$u(1) \quad \left(\frac{dy}{dx}\right)_0 + 2 - 1 = 0 \quad \left(\frac{dy}{dx}\right)_0 = -1$$

$$\text{Now } \frac{d^2 y}{dx^2} + 1 \frac{dy}{dx} - e^x = 0$$

$$\left(\frac{d^2 y}{dx^2}\right)_0 + 1(-1) - e^0 = 0 \quad \left(\frac{d^2 y}{dx^2}\right)_0 = 2$$

$$\text{Now } \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} - e^x = 0$$

$$\left(\frac{d^3 y}{dx^3}\right)_0 + 2 - 1 = 0 \quad \left(\frac{d^3 y}{dx^3}\right)_0 = -1$$

$$\textcircled{3} \text{ soln } y = 2 + x(-1) + \frac{x^2}{2}(2) + \frac{x^3}{6}(-1) + \dots$$

$$= 2 - x + x^2 - \frac{x^3}{6} + \dots$$

$$\textcircled{4} \quad \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0 \quad \text{--- (1)}$$

$$\text{given } x_0 = 0 \quad y_0 = 1 \quad \left(\frac{dy}{dx} \right)_0 = 2$$

$$\text{u(1)} \quad \left(\frac{d^2 y}{dx^2} \right)_0 + 0 + 1 = 0 \Rightarrow \left(\frac{d^2 y}{dx^2} \right)_0 = -1$$

$$\text{Now } \frac{d^3 y}{dx^3} + \left(x \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 1 \right) + 1 \frac{dy}{dx} = 0$$

$$\frac{d^3 y}{dx^3} + x \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = 0$$

$$\left(\frac{d^3 y}{dx^3} \right)_0 + 0(-1) + 2(2) = 0 \Rightarrow \left(\frac{d^3 y}{dx^3} \right)_0 = -4$$

$$\text{Now } \frac{d^4 y}{dx^4} + \left(x \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} \cdot 1 \right) + 2 \frac{d^2 y}{dx^2} = 0$$

$$\frac{d^4 y}{dx^4} + x \frac{d^3 y}{dx^3} + 3 \frac{d^2 y}{dx^2} = 0$$

$$\left(\frac{d^4 y}{dx^4} \right)_0 + 0(-4) + 3(-1) = 0 \Rightarrow \left(\frac{d^4 y}{dx^4} \right)_0 = 3$$

$$y = 1 + x(2) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-4) + \frac{x^4}{4!}(3) + \dots$$

$$y = 1 + 2x - \frac{x^2}{2} - \frac{2x^3}{3} + \frac{1}{8}x^4 + \dots$$

$$(5) \quad \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = 3xy \quad \text{--- (1)}$$

$$\text{given } x_0 = 1, y_0 = 1, \left(\frac{dy}{dx} \right)_1 = -1$$

$$\text{w① } \left(\frac{d^2 y}{dx^2} \right)_1 + 2(-1) = 3(1)(1) \Rightarrow \left(\frac{d^2 y}{dx^2} \right)_1 = 5$$

$$\text{Now } \frac{d^3 y}{dx^3} + 2 \frac{d^2 y}{dx^2} = 3x \cdot 1 \frac{dy}{dx} + y \cdot 3$$

$$\left(\frac{d^3 y}{dx^3} \right)_1 + 2(5) = 3(1)(-1) + 3(1) \Rightarrow \left(\frac{d^3 y}{dx^3} \right)_1 = -10$$

$$\text{Now } y = y_0 + (x-1) \left(\frac{dy}{dx} \right)_1 + \frac{(x-1)^2}{2!} \left(\frac{d^2 y}{dx^2} \right)_1 + \frac{(x-1)^3}{3!} \left(\frac{d^3 y}{dx^3} \right)_1 + \dots$$

$$y = 1 - (x-1) + \frac{(x-1)^2}{2} \cdot 5 + \frac{(x-1)^3}{6} \cdot -10 + \dots$$

$$y = 1 - (x-1) + \frac{5}{2}(x-1)^2 - \frac{5}{3}(x-1)^3 + \dots$$

$$(6) \quad \frac{d^2 y}{dx^2} + 2y \frac{dy}{dx} + y^3 = 1 + x \quad \text{--- (1)}$$

$$\text{given } x_0 = 0, y_0 = 1, \left(\frac{dy}{dx} \right)_0 = 1$$

$$\text{w① } \left(\frac{d^2 y}{dx^2} \right)_0 + 2(1)(1) + 1^3 = 1 + 0 \Rightarrow \left(\frac{d^2 y}{dx^2} \right)_0 = -2$$

$$\text{Now } \frac{d^3 y}{dx^3} + \left(2y \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 2 \frac{dy}{dx} \right) + 3y^2 \frac{dy}{dx} = 1$$

$$\frac{d^3 y}{dx^3} + 2y \frac{d^2 y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 + 3y^2 \frac{dy}{dx} = 1$$

$$\left(\frac{d^3 y}{dx^3} \right)_0 + 2(1)(-2) + 2(1)^2 + 3(1)^2 \cdot 1 = 1 \Rightarrow \left(\frac{d^3 y}{dx^3} \right)_0 = 0$$

$$\textcircled{6} \text{ combd } \frac{d^4 y}{dx^4} + \left(2y \cdot \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} \cdot 2 \frac{dy}{dx} \right) + \left[4 \left(\frac{dy}{dx} \right) \cdot \frac{d^2 y}{dx^2} \right] + \left[3y^2 \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 6y \frac{dy}{dx} \right] = 0$$

$$\left(\frac{d^4 y}{dx^4} \right)_0 + 2(1)0 + (-2) \cdot (2) \cdot (1) + 4(1)(-2) + 3(1)^2(-2) + (1) \cdot 6(1) \cdot 1 = 0$$

$$\left(\frac{d^4 y}{dx^4} \right)_0 = 4 + 8 + 6 - 6 = 12$$

$$\therefore y = 1 + x(1) + \frac{x^2}{2!}(-2) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(12) + \dots$$

$$y = 1 + x - x^2 + \frac{1}{2}x^4 + \dots$$

$$\textcircled{7} (1+2x) \frac{dy}{dx} = x + 2y^2 \quad \text{--- (1)}$$

$$(a) (1+2x) \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 2 = 1 + 4y \frac{dy}{dx} \quad \text{--- (2)}$$

$$(1+2x) \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} \cdot 2 + 2 \frac{dy}{dx} = 0 + 4y \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot 4 \frac{dy}{dx}$$

$$(1+2x) \frac{d^3 y}{dx^3} + 4(1-y) \frac{d^2 y}{dx^2} = 4 \left(\frac{dy}{dx} \right)^2 \quad \text{As required. --- (3)}$$

$$(b) \text{ Given } x_0=0 \quad y_0=1 \quad \text{in (1)} \quad (1+0) \left(\frac{dy}{dx} \right)_0 = 0 + 2(1)^2 \Rightarrow \left(\frac{dy}{dx} \right)_0 = 2$$

$$\text{w (2)} \quad (1+2(0)) \left(\frac{d^2 y}{dx^2} \right)_0 + 2(2) = 1 + 4(1)(2) \Rightarrow \left(\frac{d^2 y}{dx^2} \right)_0 = 5$$

$$\text{w (3)} \quad (1+2(0)) \left(\frac{d^3 y}{dx^3} \right)_0 + 4(1-1)(5) = 4(2)^2 \Rightarrow \left(\frac{d^3 y}{dx^3} \right)_0 = 16$$

$$y = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \dots$$

$$(8) \quad \sin x \frac{dy}{dx} + y \cos x = y^2$$

$$\text{Given } x_0 = \frac{\pi}{4} \quad y_0 = \sqrt{2} \quad \sin \frac{\pi}{4} \left(\frac{dy}{dx} \right)_{\frac{\pi}{4}} + \sqrt{2} \cos \frac{\pi}{4} = (\sqrt{2})^2$$

$$\frac{1}{\sqrt{2}} \frac{\sqrt{2}}{2} \left(\frac{dy}{dx} \right)_{\frac{\pi}{4}} + 1 = 2$$

$$\left(\frac{dy}{dx} \right)_{\frac{\pi}{4}} = \sqrt{2}$$

$$\text{Now } \left(\sin x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \cos x \right) + \left(y \cdot (-\sin x) + \cos x \cdot 1 \cdot \frac{dy}{dx} \right) = 2y \frac{dy}{dx}$$

$$\sin \frac{\pi}{4} \cdot \left(\frac{d^2y}{dx^2} \right)_{\frac{\pi}{4}} + \sqrt{2} \cos \frac{\pi}{4} - \sqrt{2} \cdot \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \cdot \sqrt{2} = 2\sqrt{2} \cdot \sqrt{2}$$

$$\frac{1}{\sqrt{2}} \left(\frac{d^2y}{dx^2} \right)_{\frac{\pi}{4}} + \sqrt{2} \times \frac{1}{\sqrt{2}} - \sqrt{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \sqrt{2} = \cancel{4}$$

$$\frac{1}{\sqrt{2}} \left(\frac{d^2y}{dx^2} \right)_{\frac{\pi}{4}} + 1 - 1 + 1 = \cancel{4}$$

$$\left(\frac{d^2y}{dx^2} \right)_{\frac{\pi}{4}} = 3\sqrt{2}$$

$$\therefore y = \sqrt{2} + \sqrt{2} \left(x - \frac{\pi}{4} \right) + \frac{3\sqrt{2}}{2!} \left(x - \frac{\pi}{4} \right)^2 + \dots$$

$$y = \sqrt{2} + \sqrt{2} \left(x - \frac{\pi}{4} \right) + \frac{3\sqrt{2}}{2} \left(x - \frac{\pi}{4} \right)^2 + \dots$$

$$(9) \quad \frac{dy}{dx} - x^2 - y^2 = 0 \quad \text{--- (1)}$$

$$(a) \quad (i) \quad \frac{d^2y}{dx^2} - 2x - 2y \frac{dy}{dx} = 0 \quad \text{--- (2)}$$

$$(ii) \quad \frac{d^3y}{dx^3} - 2 - \left[2y \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2 \cdot \frac{dy}{dx} \right] = 0$$

$$\frac{d^3y}{dx^3} - 2y \frac{d^2y}{dx^2} - 2 \left(\frac{dy}{dx} \right)^2 = 2 \quad \text{--- (3)}$$

$$(b) \quad \frac{d^4y}{dx^4} - \left[2y \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} \cdot 2 \frac{dy}{dx} \right] - 4 \left(\frac{dy}{dx} \right) \cdot \frac{d^2y}{dx^2} = 0$$

$$\frac{d^4y}{dx^4} - 2y \frac{d^3y}{dx^3} - 6 \left(\frac{dy}{dx} \right) \left(\frac{d^2y}{dx^2} \right) = 0 \quad \text{--- (4)}$$

$$(c) \quad x_0 = 0 \quad y_0 = 1$$

$$u(1) \quad \left(\frac{dy}{dx} \right)_0 - 0^2 - 1^2 = 0 \quad \Rightarrow \quad \left(\frac{dy}{dx} \right)_0 = 1$$

$$u(2) \quad \left(\frac{d^2y}{dx^2} \right)_0 - 2(0) - 2(1)(1) = 0 \quad \Rightarrow \quad \left(\frac{d^2y}{dx^2} \right)_0 = 2$$

$$u(3) \quad \left(\frac{d^3y}{dx^3} \right)_0 - 2(1)(2) - 2(1)^2 = 2 \quad \Rightarrow \quad \left(\frac{d^3y}{dx^3} \right)_0 = 8$$

$$u(4) \quad \left(\frac{d^4y}{dx^4} \right)_0 - 2(1)(8) - 6(1)(2) = 0 \quad \Rightarrow \quad \left(\frac{d^4y}{dx^4} \right)_0 = 28$$

$$\therefore y = 1 + x(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(8) + \frac{x^4}{4!}(28) + \dots$$

$$y = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \dots$$

$$(10) \quad \cos x \frac{dy}{dx} + y \sin x + 2y^3 = 0 \quad \text{--- (1)}$$

$$\left[\cos x \cdot \frac{dy}{dx} + \frac{dy}{dx} \cdot \sin x \right] + \left[y \cdot \cos x + \sin x \cdot 1 \cdot \frac{dy}{dx} \right] + 6y^2 \frac{dy}{dx} = 0$$

$$\cos x \frac{dy}{dx} + 6y^2 \frac{dy}{dx} + y \cos x = 0 \quad \text{--- (2)}$$

$$\left[\cos x \cdot \frac{dy}{dx} + \frac{dy}{dx} \cdot \sin x \right] + \left[6y^2 \cdot \frac{dy}{dx} + \frac{dy}{dx} \cdot 12y \frac{dy}{dx} \right] + \left[y \cdot \sin x + \cos x \cdot 1 \cdot \frac{dy}{dx} \right] = 0$$

--- (3)

Now given $x_0 = 0$ $y_0 = 1$

$$w1) \quad \cos 0 \left(\frac{dy}{dx} \right)_0 + (1) \sin 0 + 2(1)^3 = 0 \Rightarrow \left(\frac{dy}{dx} \right)_0 = -2$$

$$w2) \quad \cos 0 \left(\frac{d^2y}{dx^2} \right)_0 + 6(1)^2(-2) + (1) \cos 0 = 0 \Rightarrow \left(\frac{d^2y}{dx^2} \right)_0 = 11$$

$$w3) \quad \cos 0 \left(\frac{d^3y}{dx^3} \right)_0 + (11)(-\sin 0) + 6(1)^2(11) + (-2)(12)(1)(-2) + (1)(0) + (1)(1)(-2) = 0$$

$$\Rightarrow \left(\frac{d^3y}{dx^3} \right)_0 + 66 + 48 - 2 = 0 \Rightarrow \left(\frac{d^3y}{dx^3} \right)_0 = -112$$

$$\therefore y = 1 - 2x + \frac{11x^2}{2!} - \frac{112x^3}{6} + \dots$$

$$y = 1 - 2x + \frac{11x^2}{2!} - \frac{56x^3}{3} + \dots$$