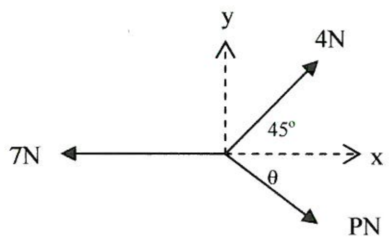
Using Vectors in Mechanics

Often vectors can be used in mechanics to simplify how a situation is described, or how a calculation is performed.

In a nutshell, a vector quantity has already been split into its x and y component parts. Consider the following equilibrium question:

The following system of three forces is in equilibrium. Find the value of P and the angle θ.



$\Sigma F_x: 4\cos 45 + P\cos\theta - 7 = 0 \Rightarrow P\cos\theta = 7 - 4\cos 45 \quad — (1)$

$\Sigma F_y: 4\sin 45 - P\sin\theta = 0 \Rightarrow P\sin\theta = 4\sin 45 \quad — (2)$

$(2) \div (1) \quad \tan\theta = \dfrac{4\sin 45}{7 - 4\cos 45} \Rightarrow \theta = 34.1^\circ \quad$ in (2) $P = \dfrac{4\sin 45}{\sin 34.1} = 5.0N$

Now this question could have been given as a vector problem:

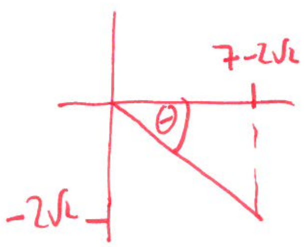
Three forces act on a particle which remains in equilibrium. If two of the forces are $\mathbf{F}_1 = (2\sqrt{2}i + 2\sqrt{2}j)$ and $\mathbf{F}_2 = -7i$, find the magnitude of the third force and the direction in which it acts.

$F_1 + F_2 + F_3 = 0$

$(2\sqrt{2}\underline{i} + 2\sqrt{2}\underline{j}) + (-7\underline{i}) + F_3 = 0$

$F_3 = (7 - 2\sqrt{2})\underline{i} - 2\sqrt{2}\underline{j}$

$|F_3| = \sqrt{(7-2\sqrt{2})^2 + (-2\sqrt{2})^2} = 5.0N$



$\theta = \tan^{-1}\left(\dfrac{2\sqrt{2}}{7-2\sqrt{2}}\right) = 34.1^\circ$ below x axis

Forces are vector quantities as they have magnitude and direction. Velocity and acceleration are also vector quantities. The magnitude of velocity is its *speed.* Mass is not a vector quantity – it is known as a scalar quantity as is time.

Questions involving constant acceleration formulae and Newtons Laws can also be approached using vectors.

<u>Eg1</u> Find the magnitude of the acceleration of a body of mass 500g when forces of $(5\mathbf{i} + 3\mathbf{j})$N, $(6\mathbf{i} + 4\mathbf{j})$N and $(-7\mathbf{i} - 7\mathbf{j})$N act on the body.

$$(5i+3j)+(6i+4j)+(-7i-7j) = 0.5\underline{a}$$

$$4\underline{i} = 0.5a$$

$$a = 8i \text{ ms}^{-2}$$

$$|a| = \sqrt{8^2+0^2} = 8 \text{ ms}^{-2}$$

<u>Eg2</u> A particle has an initial velocity of $(2i + 3j)\text{ms}^{-1}$ and 5 seconds later, a velocity of $(5i – j)\text{ms}^{-1}$. Find its acceleration, assumed constant, and the magnitude of that acceleration.

$$u = (2i+3j) \quad v = (5i-j), \; t = 5, \; a = ?$$

$$\underline{v} = \underline{u} + \underline{a}t$$

$$5i - j = 2i + 3j + 5\underline{a}$$

$$3i - 4j = 5\underline{a}$$

$$\underline{a} = 0.6i - 0.8j$$

$$|a| = \sqrt{0.6^2+0.8^2} = 1 \text{ ms}^{-2}$$

<u>Vectors PPQ's</u> Leave the 2nd one until last.

M1 – Vectors PPQ's (Low Level)

M1 Jan 2009

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1. A particle P moves with constant acceleration $(2\mathbf{i} - 5\mathbf{j})$ m s^{-2}. At time $t = 0$, P has speed u m s^{-1}. At time $t = 3$ s, P has velocity $(-6\mathbf{i} + \mathbf{j})$ m s^{-1}.

   Find the value of u.

   (5)

M1 - June 09

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2. A particle is acted upon by two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, given by

   $\mathbf{F}_1 = (\mathbf{i} - 3\mathbf{j})$ N,

   $\mathbf{F}_2 = (p\mathbf{i} + 2p\mathbf{j})$ N, where p is a positive constant.

   (a) Find the angle between $\mathbf{F}_2$ and $\mathbf{j}$.

   (2)

   The resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ is $\mathbf{R}$. Given that $\mathbf{R}$ is parallel to $\mathbf{i}$,

   (b) find the value of p.

   (4)

M1 Jan 07

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3. A particle P of mass 2 kg is moving under the action of a constant force $\mathbf{F}$ newtons. When $t = 0$, P has velocity $(3\mathbf{i} + 2\mathbf{j})$ m s^{-1} and at time $t = 4$ s, P has velocity $(15\mathbf{i} - 4\mathbf{j})$ m s^{-1}. Find

   (a) the acceleration of P in terms of $\mathbf{i}$ and $\mathbf{j}$,

   (2)

   (b) the magnitude of $\mathbf{F}$,

   (4)

   (c) the velocity of P at time $t = 6$ s.

   (3)

M1 Jan 07

3. A particle P of mass 0.4 kg is moving under the action of a constant force $\mathbf{F}$ newtons. Initially the velocity of P is $(6\mathbf{i} - 27\mathbf{j})$ m s^{-1} and 4 s later the velocity of P is $(-14\mathbf{i} + 21\mathbf{j})$ m s^{-1}.

   (*a*) Find, in terms of $\mathbf{i}$ and $\mathbf{j}$, the acceleration of P.

   (3)

   (*b*) Calculate the magnitude of $\mathbf{F}$.

   (3)

M1 June 09

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3. A particle P of mass 0.4 kg moves under the action of a single constant force $\mathbf{F}$ newtons. The acceleration of P is $(6\mathbf{i} + 8\mathbf{j})$ m s^{-2}. Find

   (a) the angle between the acceleration and $\mathbf{i}$,

   (2)

   (b) the magnitude of $\mathbf{F}$.

   (3)

   At time t seconds the velocity of P is $\mathbf{v}$ m s^{-1}. Given that when $t = 0$, $\mathbf{v} = 9\mathbf{i} - 10\mathbf{j}$,

   (c) find the velocity of P when $t = 5$.

   (3)

M1 Jan 2009

Q1 $\underline{a} = 2i - 5j$, $t = 3$, $\underline{v} = (-6i + j)$.

$\underline{v} = \underline{u} + at$

$(-6i + j) = u + 3(2i - 5j)$

$u = -6i + j - 6i + 15j$

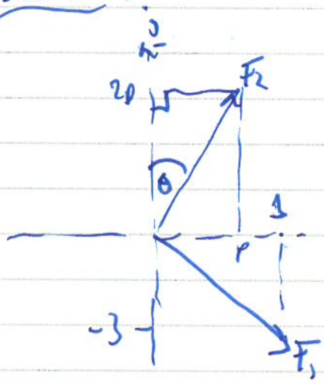
$u = -12\underline{i} + 16\underline{j}$

$|u| = \sqrt{(-12)^2 + 16^2} = 20\,ms^{-1}$

M1 June 09

Q2

(a)



$\tan\theta = \cancel{2p}\ \frac{p}{2p}$

$\theta = \tan^{-1}(\tfrac{1}{2}) = \cancel{63.4}\ 26.6^\circ$

(b) $F_1 + F_2 = R$

$i - 3j + \cancel{2}pi + 2p\underline{j} = R$

$R = (1+p)i + (2p - 3)j$

if R is parallel to i then $\underline{j}$ component $= 0$ $\therefore$ $2p - 3 = 0$

$p = \frac{3}{2}$

M1 - Jan 07

Q3 (a) $u = 3i + 2j$, $t = 4$ $v = 15i - 4j$, $a = ?$

$\underline{v} = \underline{u} + \underline{a}t$

$15i - 4j = 3i + 2j + 4a$

$4a = 12i - 6j$

$a = 3i - 1.5j$

(b) $F = ma$

$F = 2(3i - 1.5j)$

$F = 6i - 3j$

$|F| = \sqrt{36 + 9} = 3\sqrt{5}\,N$

(c) @ $t = 6$ $\underline{v} = \underline{u} + \underline{a}t$

$\underline{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1.5 \end{pmatrix} . 6$

$\underline{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 18 \\ -9 \end{pmatrix}$

$\underline{v} = \begin{pmatrix} 21 \\ -7 \end{pmatrix} = 21i - 7j \ ms^{-1}$

M1 – JAN 03

Q3 $u = (6i - 27j)$ $t = 4$ $V = (-14i + 21j)$ $a = ?$

(a) $V = u + at$

$$\begin{pmatrix}-14\\21\end{pmatrix} = \begin{pmatrix}6\\-27\end{pmatrix} + 4a$$

$$\begin{pmatrix}-20\\48\end{pmatrix} = 4a$$

$$a = \begin{pmatrix}-5\\12\end{pmatrix} = -5i + 12j$$

(b) $F = ma$

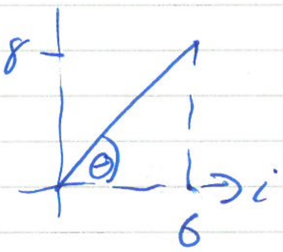
$F = 0.4(-5i + 12j) = (-2i + 4.8j)$

$|F| = \sqrt{4 + 23.04} = 5.2N.$

M1 – JUNE 08

Q3

(a)



$\tan\theta = \frac{8}{6}$

$\theta = 53.1°.$

(b) $F = 0.4(6i + 8j) = 2.4i + 3.2j$

$|F| = \sqrt{2.4^2 + 3.2^2} = 4N$

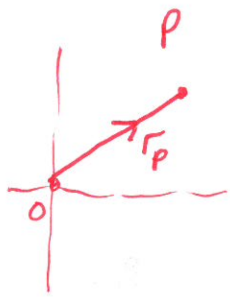
(c) $u = 9i - 10j$, $t = 5$, $a = (6i + 8j)$, $V = ?$

$V = (9i - 10j) + 5(6i + 8j)$

$= 9i - 10j + 30i + 40j$

$= 39i + 30j \ ms^{-1}$

Position Vectors

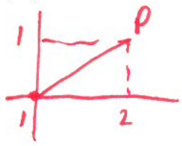


Imagine a particle P moving in a plane. O is a fixed point in the plane. If you know where the point O is, then the position of P is uniquely defined by the vector:

$$\overrightarrow{OP} = \underline{r}_p$$

The vector $\mathbf{r_p}$ is called the *position vector* of P relative to O.

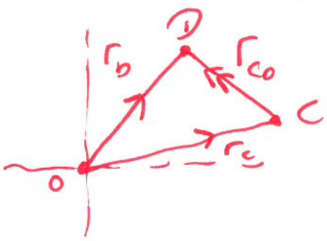
<u>Eg3</u> At a given time the cartesian co-ordinates of the position P of a particle are (2,1). Find the position vector of P relative to O.



$$r_p = 2\underline{i} + \underline{j}$$

Relative Position Vector

Imagine two particles C and D moving in a plane; O is a fixed point in the plane. Then $\mathbf{r_c}$ is the position vector of C relative to O, and $\mathbf{r_d}$ is the position vector of D relative to O.



The vector $\overrightarrow{CD}$ gives the position vector of D relative to C. It is called the *relative position vector.*

From the triangle law of addition $\overrightarrow{CD} = -\overrightarrow{OC} + \overrightarrow{OD}$

$$r_{CD} = r_D - r_C$$

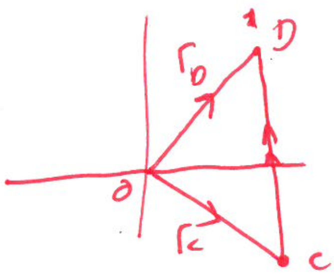
- **The position vector of D relative to** $r_{\overrightarrow{CD}} = r_D - r_C$

The position vector of C relative to D is

DC = -CD = $\mathbf{r_c} - \mathbf{r_d}$

the negative of the position vector of D relative to C.

<u>Eg4</u> At a given time particle C has position vector (4**i** – 6**j**)m relative to a fixed origin O and particle D has position vector (3**i** + 2**j**)m relative to O. Find the position vector of D relative to C.



$$r_{CD} = -r_C + r_D$$

$$= -\begin{pmatrix} 4 \\ -6 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

Velocity as a Vector

If a particle is moving at a *speed* of $5ms^{-1}$ then, provided its speed remains constant, it will travel 5m in every second. This is true no matter what type of path the particle is moving on. It will travel 10m in 2s, 15m in 3s and so on. So in this case:

distance travelled = speed x time

But if you want a complete picture of what is happening, you also need to know the *direction* in which the particle is moving. The *velocity* of the particle gives us a complete picture.

- **The velocity of a particle is a vector in the direction of motion whose magnitude is equal to the speed of the particle.**

It is usually denoted by **v**.

When distances are measured in metres and time in seconds, velocities are measured in metres per second (ms^{-1}).

If the velocity is a constant vector then:

displacement = velocity x time

If, at t = 0 the particle is at the origin, then t seconds later, the particle has position vector

$$\underline{r}_t = \underline{v}t$$

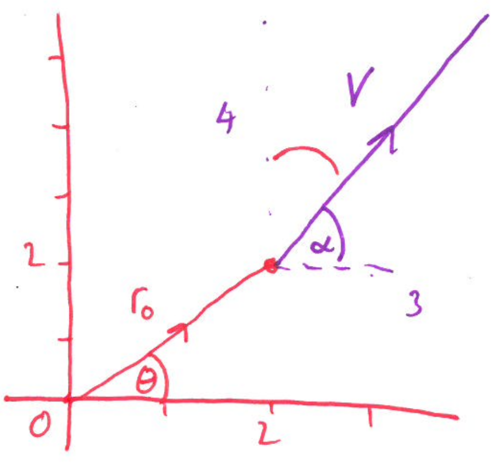
However if the particle is at a position vector $\mathbf{r_o}$ at t = 0, then t seconds later, the particle has a position vector

$$\underline{r}_t = \underline{r}_0 + \underline{v}t$$

The magnitude of a position vector will give the distance of the particle from the origin, the direction of a position vector will give the direction of the particle relative to the origin.

The magnitude of a velocity vector will give the speed of the particle and the direction of a velocity vector will give the direction in which the particle is traveling.

Consider a particle traveling with a constant velocity $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ with an initial position $\mathbf{r_o} = 2\mathbf{i} + 2\mathbf{j}$



$|r_0| = \sqrt{2^2+2^2} = 2\sqrt{2}$

$\theta = 45°$

ie particle initially $2\sqrt{2}$m from origin at angle 45° above ~~origin~~ x axis.

$|v| = \sqrt{3^2+4^2} = 5$

$\alpha = \tan^{-1}(\frac{4}{3}) = 53.1°$

Moving with speed $5ms^{-1}$ on a bearing 037°.

Acceleration as a Vector
Everybody has some idea of acceleration form their experience of travelling in busses or cars. Just as velocity tells you how the position of a particle changes with time, the *acceleration* tells you how the velocity changes with time. Since velocity has magnitude and direction so has acceleration. It is, therefore, a vector. It is usually denoted by **a**.

- **acceleration is the rate of change of velocity with respect to time.**

Hence: $$\text{acceleration} = \frac{\text{change in velocity}}{\text{time}}$$
when the acceleration is constant.

Since velocities are measured in metres per second (ms^{-1}) the acceleration is measured in metres per second per second (ms^{-2}).

Eg5 A particle P is moving with a constant velocity $(12\mathbf{i} + 5\mathbf{j})ms^{-1}$. It passes through the point A whose position vector is $(4\mathbf{i} + 5\mathbf{j})$m at $t = 0$. Find:

(a) the speed of the particle

(b) the distance of P from O when $t = 3$s.

Eg6 At noon a lighthouse keeper observes two ships A and B which have position vectors $(-4\mathbf{i} + 3\mathbf{j})$km and $(4\mathbf{i} + 9\mathbf{j})$km respectively, relative to the lighthouse O. (The unit vectors **i** and **j** are directed due east and due north.) The ships are moving with constant velocities $(4\mathbf{i} + 17\mathbf{j})kmh^{-1}$ and $(-12\mathbf{i} + 5\mathbf{j})kmh^{-1}$ respectively.

(a) Write down the position vector of A and the position vector of B at time t hours after noon.

(b) Show that A and B will collide and find the time when this collision will occur and the position vector of the point of collision.

Exercise 6F Pg 148 Q's 9 to 12

Eg5. $r_0 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ $V = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$

(a) $|V| = \sqrt{12^2 + 5^2} = 13\,ms^{-1}$

(b) $r_t = r_0 + Vt$

when $t=3$ $r_3 = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 12 \\ 5 \end{pmatrix} . 3 = \begin{pmatrix} 40 \\ 20 \end{pmatrix}$

$\therefore$ distance from O $= \sqrt{40^2 + 20^2} = 44.7$ metres

Eg6(a) Ship A: $r_0 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ $V = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$

$\therefore$ $r_A = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 17 \end{pmatrix} t = \begin{pmatrix} 4t - 4 \\ 17t + 3 \end{pmatrix}$ — 1

(b) Ship B: $r_0 = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$ $V = \begin{pmatrix} -12 \\ 5 \end{pmatrix}$

$\therefore$ $r_B = \begin{pmatrix} 4 \\ 9 \end{pmatrix} + \begin{pmatrix} -12 \\ 5 \end{pmatrix} t = \begin{pmatrix} 4 - 12t \\ 5t + 9 \end{pmatrix}$

(b) For ships to collide $r_A = r_B$ for same value of t

ie $4 - 12t = 4t - 4$

$16t = 8$

$t = \frac{1}{2}$

& $17t + 3 = 5t + 9$

$12t = 6$

$t = \frac{1}{2}$

$\therefore$ ships will collide at 12:30

in 1 $r_a = \begin{pmatrix} 2 - 4 \\ 17/2 + 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 11.5 \end{pmatrix}$

Eg6 (b) Alternative: pos vect of A relative to B

$$r_{\overrightarrow{BA}} = -r_B + r_A$$

$$= -\begin{pmatrix} 4-12t \\ 5t+9 \end{pmatrix} + \begin{pmatrix} 4t-4 \\ 17t+3 \end{pmatrix}$$

$$= \begin{pmatrix} 16t-8 \\ 12t-6 \end{pmatrix}$$

Now if ships collide then $r_{\overrightarrow{BA}} = 0$

$\therefore$ $16t - 8 = 0$ & $12t - 6 = 0$

$t = \frac{1}{2}$ $\qquad$ $t = \frac{1}{2}$

(8) P: $r_0 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ $\quad V = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Q: $r_0 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ $\quad V = \underline{V}$

$r_p = r_0 + Vt \qquad r_p = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r_Q = \begin{pmatrix} 0 \\ -3 \end{pmatrix} + Vt$

when $t = 8$ $\quad r_p = r_Q$

$$\begin{pmatrix} 4+t \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} + Vt$$

$$\begin{pmatrix} 12 \\ 8 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 8V$$

$$\begin{pmatrix} 12 \\ 11 \end{pmatrix} = 8V$$

$$V = \begin{pmatrix} 12/8 \\ 11/8 \end{pmatrix}$$

Speed of Q $= \sqrt{\frac{144}{64} + \frac{121}{64}} = 2.03\ ms^{-1}$

(9) (a) $r = \begin{pmatrix} 0 \\ -500 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} t = \begin{pmatrix} 2t \\ 3t-500 \end{pmatrix}$

(b) $t = 300 \quad r = \begin{pmatrix} 600 \\ 400 \end{pmatrix}$

dist $= \sqrt{600^2 + 400^2} = 721$ metres

(10) (a) F: $r_F = \begin{pmatrix} 0 \\ 400 \end{pmatrix} + \begin{pmatrix} 7 \\ 7 \end{pmatrix} t = \begin{pmatrix} 7t \\ 7t+400 \end{pmatrix}$

S: $r_S = \begin{pmatrix} 500 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 15 \end{pmatrix} t = \begin{pmatrix} 500-3t \\ 15t \end{pmatrix}$

If collision then $r_F = r_S$ for same value of t

Compare components i: $7t = 500 - 3t$

$10t = 500$

$t = 50s.$

j: $7t + 400 = 15t$

$400 = 8t$

$t = 50s$

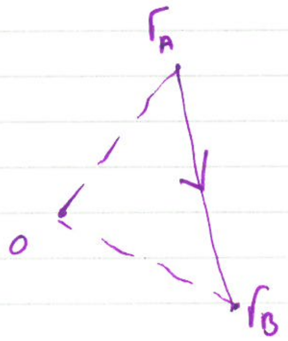
(b) $\therefore$ collision occurs @ 50s.

$r = \begin{pmatrix} 350 \\ 750 \end{pmatrix}$

11. (a) A: $r = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix}t = \begin{pmatrix}2t+1\\3-t\end{pmatrix}$

B: $r = \begin{pmatrix}5\\-2\end{pmatrix} + \begin{pmatrix}-1\\4\end{pmatrix}t = \begin{pmatrix}5-t\\4t-2\end{pmatrix}$

(b)



r_{BA} = position of B relative to A

means what vector to get to B from A

$= -r_A + r_B$

$= -\begin{pmatrix}2t+1\\3-t\end{pmatrix} + \begin{pmatrix}5-t\\4t-2\end{pmatrix}$

$r_{BA} = \begin{pmatrix}4-3t\\5t-5\end{pmatrix}$

(c) If ships collide $r_{BA} = 0$ i: $4-3t=0$

$t = \frac{4}{3}$

j: $5t-5=0$

$t=1$

this occurs @ different times ∴ $r_B \neq 0$ so ships don't collide

(d) When $t=2$ $r_{BA} = \begin{pmatrix}-2\\5\end{pmatrix}$

distance between ships $\sqrt{4+25} = \sqrt{29} = 5.39$ km

(12) (a) $V_A = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} t = \begin{pmatrix} 2t-1 \\ 1-4t \end{pmatrix}$

$V_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} t = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$

when $t=3$ $V_A = \begin{pmatrix} 5 \\ -11 \end{pmatrix}$ $|V_A| = \sqrt{25+121} = 12.1 \, ms^{-1}$

$V_B = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ $|V_B| = \sqrt{37} = 6.08 \, ms^{-1}$

(b) $r_A = \begin{pmatrix} 12 \\ 12 \end{pmatrix} + \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} t + \frac{1}{2} \begin{pmatrix} 2 \\ -4 \end{pmatrix} t^2 \right]$

$r_A = \begin{pmatrix} t^2 - t + 12 \\ -2t^2 + t + 12 \end{pmatrix}$

@ $t=3$ $r_A = \begin{pmatrix} 9 - 3 + 12 \\ -18 + 3 + 12 \end{pmatrix} = \begin{pmatrix} 18 \\ -3 \end{pmatrix}$

(c) Now $r_B = \begin{pmatrix} r_0 \end{pmatrix} + \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \frac{1}{2} \begin{pmatrix} 0 \\ 2 \end{pmatrix} t^2 \right]$

but @ collision $r_B = \begin{pmatrix} 18 \\ -3 \end{pmatrix}$

$\begin{pmatrix} 18 \\ -3 \end{pmatrix} = r_0 + \begin{pmatrix} t \\ t^2 \end{pmatrix}$

@ $t=3$ $\begin{pmatrix} 18 \\ -3 \end{pmatrix} = r_0 + \begin{pmatrix} 3 \\ 9 \end{pmatrix}$

$r_0 = \begin{pmatrix} 15 \\ -12 \end{pmatrix}$